OT REPORT 77-118

CONFIDENCE LIMITS FOR DIGITAL ERROR RATES FROM DEPENDENT TRANSMISSIONS

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U.S. DEPARTMENT OF COMMERCE / Office of Telecommunications

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March 1977

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CONFIDENCE LIMITS FOR DIGITAL ERROR RATES FROM DEPENDENT TRANSMISSIONS

Edwin L. Crow and Martin J. Miles*

Approximate confidence limits for error rates (probabilities of an error) of digital communication systems are derived under the assumptions that the sample is large, the error rate is constant and the dependence between transmissions is governed by a stationary first-order Markov chain model. Five different approximations are given: normal distribution, Edgeworth two-term and four-term, Pearson system, and modified Poisson-Anderson-Burstein. Results by Gabriel and by Klotz are used. The limits are compared with Ladd's small-sample limits and illustrated on telephone data. Methods for testing the validity of the model are described. A computer program for applying the limits is presented, and an extensive Monte Carlo simulation to test the accuracy of the limits is summarized. Methods for designing the experiment so as to achieve a specified precision are given.

Key words: Bernoulli trials, chi-squared test, confidence limits, confidence region, design of experiments, digital communication systems, Edgeworth series, Freeman-Tukey deviates, likelihood ratio test, Markov chain, Monte Carlo simulation, nonstationarity, Pearson system

1. INTRODUCTION

The successive transmissions of bits or words by a communication system show dependence in the incidence of errors, error "bursts" for example. Hence the classical Bernoulli model of independent trials with constant error rate (probability of error) is inadequate for calculating confidence limits or "error bars" for the error rate. One would expect the error

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bars calculated from the Bernoulli model to be shorter than the correct error bars due to the dependence. The purpose of the present report is to obtain realistic confidence limits by taking first-order account of the dependence between successive transmissions. This is done by introducing just one additional parameter, the conditional probability of an error given that an error has just occurred. The resulting model is a stationary Markov chain with two states.

Gabriel (1959) and Klotz (1973) derived many properties of this model: the exact distribution and moments of the number of errors ("successes"), sufficient statistics, estimates of the parameters, and their large-sample distributions. Ladd (1975) published a small-sample algorithm and tables recently. For communications applications the large-sample distributions are what are required. Exact results are impractical and unnecessary. Hence five different approximations to confidence limits for the error rate are derived herein based on the work of Gabriel and Klotz.

Finite Markov chains were introduced by A.A. Markov in 1907 (Cox and Miller, 1965, p. 141) and have been widely applied. Many more complicated mathematical models have been suggested for communication systems. Perhaps the first and simplest to take account of dependence is that of Gilbert (1960). He assumed that the system could have two different states, a good one, G, that is error-free and a bad one, B, that has a non-zero probability of producing an error. The model has three parameters, the error probability when it is in state B and the two transition probabilities, from G to B and from B to G.

Elliott (1963) generalized Gilbert's model to allow a non-zero error probability in the good state also. Subsequent authors (cf. Fritchman, 1967) have used six or more parameters for the purpose of describing the data more completely. The models have been applied to probabilistic description of digital data transmission, in particular to the evaluation of codes.

To introduce the model assumed throughout this report, we follow Klotz (1973) and consider a sequence of identically distributed random variables $X_1, X_2, \ldots, X_n$ each of which can take on just two values: 1, corresponding to an error, and 0, corresponding to a correct transmission (the two states of our Markov chain). The model has two constant parameters,

$$p = P[X_i=1] = 1 - P[X_i=0], \quad i=1,2,\ldots,n, \tag{1.1}$$

$$\lambda = P[X_i=1 | X_{i-1}=1], \quad i=2,3,\ldots,n, \tag{1.2}$$

where P[ ] denotes the probability of the indicated random event and P [|] denotes the conditional probability of the event indicated first, given that the event indicated second does occur. By definition

$$P[X_i=j | X_{i-1}=k] = \frac{P[(X_{i-1}=j)\&(X_i=k)]}{P[X_{i-1}=k]}, \quad j=0,1;k=0,1. \tag{1.3}$$

The constancy of p and λ means that the chain is stationary, and the condition that λ is made to depend only on the immediately preceding transmission (trial) is the defining Markov chain property. Since there are only two states, it follows from (1.2) that

$$P[X_i=0 | X_{i-1}=1] = 1 - \lambda. \tag{1.4}$$

Just as in (independent) Bernoulli trials we let q=1-p. Since

$$P[X_i=1] = P[(X_i=1)\&(X_{i-1}=1)] + P[(X_i=1)\&(X_{i-1}=0)],$$

$$P[X_i=0] = P[(X_i=0)\&(X_{i-1}=1)] + P[(X_i=0)\&(X_{i-1}=0)],$$

it follows from (1.1)-(1.3) that

$$P[X_i=1 | X_{i-1}=0] = (1-\lambda)p/q, \tag{1.5}$$

$$P[X_i=0 | X_{i-1}=0] = (1-2p+\lambda p)/q. \tag{1.6}$$

Since all of the probabilities are between 0 and 1, the parameters are restricted by $0 \le p \le 1$ and $\max(0, 2-1/p) \le \lambda \le 1$. Furthermore the probability properties of an entire sample of n transmissions are determined by p, λ, and n. We see that

$\lambda = p$ means the transmissions are independent,

$\lambda > p$ means clustering of 1's and 0's, and

$\lambda < p$ means 1's and 0's tend to alternate.

It can be shown that the above stationary two-state Markov model is the special case of Gilbert's (1960) model in which his two transition probabilities are equal. Although it is therefore a very specialized model, the parameter λ does take dependence into account and appears to permit realistic approximation to confidence limits for p, as is argued further in section 10.1.

Section 2 states some of Klotz's (1973) results and applies them to deriving confidence limits for p and for λ and a joint confidence region for (λ,p) based on the asymptotic (large-sample) normal distribution of the estimates $(\hat{\lambda},\hat{p})$. Section 3 obtains presumably a better approximation to the confidence limits for p by using a two-term approximation for the distribution of $\hat{p}$, the first term being the normal distribution, the second being proportional to the asymmetry (skewness) measure provided by the third central moment of $\hat{p}$. Section 4 gets limits by extending the same Edgeworth series to four terms using the fourth moment. Four-moment alternative limits are provided in Section 5 using the Pearson system of distribution curves. The simplest approximate limits of all are suggested in Section 6, simply a lengthening (by a ratio of standard deviations) of the Anderson-Burstein (1967, 1968; see also Crow, 1974, 1975; and Crow and Miles, 1976) refinement of the Poisson approximation to binomial confidence limits.

These approximate confidence limits are compared in Section 7 for small samples with Ladd's (1975) limits, which are exact if λ is known. Their use is illustrated by application to some real telephone data (Cox and Lewis, 1966) in Section 9. In Section 10 it is shown how the assumption of the Markov chain model can be tested (as well as the assumption of independent transmissions). Also in Section 10 is a test for abrupt changes in regime, that is, abrupt changes in (λ,p) at specified times. These tests of the assumptions are illustrated in detail on the Cox-Lewis data.

A computer program for the five pairs of confidence limits for p is reproduced in Appendix B and discussed in Section 8.

The validity of the approximations was tested by Monte Carlo simulation of 1000 samples for a range of parameter values: $.0003 \le p \le .5$; $\lambda = .3, .8$; $50 \le n \le 1000$; confidence level = 90%, 95%. A complex program was developed for performing the simulations and recording an extensive summary of the results of each case. These summaries are reproduced in Appendix A and discussed in Section 11.

All of the above material concerns the analysis of given data, i.e., a given sequence of errors and correct transmissions. Section 12 considers the prior problem of designing an experiment to determine the error rate p and the conditional error rate λ with a specified precision or a specified budget. Numerical examples are given.

Broad conclusions to the entire study are stated in Section 13. Section 15 comprises a substantial list of references; it does not constitute a complete bibliography.

2. CONFIDENCE INTERVALS AND REGIONS FROM THE NORMAL APPROXIMATION

Klotz (1973) has derived estimators of p and λ that are consistent and asymptotically normally distributed, and he gives their asymptotic variance-covariance matrix. They are the sample mean

$$\hat{p} = \bar{x} = s/n \tag{2.1}$$

and

$$\hat{\lambda} = \hat{\lambda}(\hat{p}) = \tfrac{1}{2}(s-\hat{p})^{-1}[r-s+t+(2s-t-1)\hat{p} + \{[r-s+t+(2s-t-1)\hat{p}]^2 + 4r(s-\hat{p})(1-2\hat{p})\}^{\frac{1}{2}}], \tag{2.2}$$

where n is the sample size (number of transmissions or trials),

$$r = \sum_{i=2}^{n} x_{i-1}x_i, \quad s = \sum_{i=1}^{n} x_i, \quad t = x_1+x_n, \tag{2.3}$$

and $x_i=1$ if the ith trial results in a success, $x_i=0$ if it results in a failure. (There are slight changes from Klotz's notation and form.) If $s=0$, then

$$r = t = \hat{p} = 0,$$

and $\hat{\lambda}$ is indeterminate.

An intuitive estimator of λ is essentially the relative frequency,

$$\lambda^* = \frac{r}{s-\hat{p}} = \frac{nr}{(n-1)s} \quad (s>0). \tag{2.4}$$

[After this report was essentially completed, Devore (1976) published "modified maximum likelihood" estimates of p and λ that differ slightly in effect from (2.1), (2.2), and (2.4) and the detailed properties of which it would be of interest to investigate.]

Following the pattern of Pearson and Hartley (1966, Sec. 22.2), we seek a central $100(1-2\alpha)$ percent confidence interval for p by determining two values of p, p_L and p_U, such that

$$\sum_{i=s}^{n} f(i|p_L,\lambda,n) = \alpha, \quad \sum_{i=0}^{s} f(i|p_U,\lambda,n) = \alpha, \tag{2.5}$$

where $f(i|p,\lambda,n)$ denotes the probability function of the random variable $s=\sum_{i=1}^{n} x_i$ with parameters p, λ, and n. Here λ is unknown, a nuisance parameter, and we replace it by $\hat{\lambda}$ from (2.2) or λ^* from (2.4), relying on a sufficiently large n and the consistency of $\hat{\lambda}$ to produce about as many intervals that fail to include p by being too short as ones that include p by being too long.

To obtain a normal approximation confidence interval for p, we replace the sums in (2.5) by a normal integral with mean and standard deviation those of s with an appropriate p value. The mean of s is np. The variance of s, given by Klotz (1973) (with an incorrect exponent n rather than n-1) and by Gabriel (1959), can be written as

$$\text{Var } s = \sigma_s^2 = npq + \frac{2pq\rho}{1-\rho}\left(n-\frac{1-\rho^n}{1-\rho}\right), \quad |\rho|<1, \tag{2.6}$$

where $\rho=(\lambda-p)/q$ is the correlation coefficient between x_{i-1} and x_i. For large n and ρ bounded from 1, the term ρ^n can be neglected (e.g., $0.9^{50}=0.005$), so that

$$\sigma_s^2 \doteq \frac{npq}{(1-\rho)^2}\left(1-\rho^2-\frac{2\rho}{n}\right). \tag{2.7}$$

We assume $0<p\leq 0.5$ from this point on, including later sections. If, further, p is assumed to be small relative to q and λ and confidence limits for p are sought, then the coefficient of p on the right-hand side of (2.7) can be estimated from the sample by

$$\hat{V} = \frac{n\hat{q}}{(1-\hat{\rho})^2}\left(1-\hat{\rho}^2-\frac{2\hat{\rho}}{n}\right) \tag{2.8a}$$

where

$$\hat{\rho} = (\hat{\lambda}-\hat{p})/\hat{q}, \text{ for } |\hat{\rho}|<1. \tag{2.8b}$$

We let u_α denote the upper 100α percentage point of the standardized normal distribution, so that

$$\int_{u_\alpha}^{\infty} (2\pi)^{-\frac{1}{2}} e^{-x^2/2} dx = \alpha. \tag{2.9}$$

The normal approximation of the first left-hand sum in (2.5) is

$$\int_{s-\frac{1}{2}}^{\infty} (2\pi)^{-\frac{1}{2}}\sigma_s^{-1} e^{-(x-np_L)^2/(2\sigma_s^2)} dx = \int_{\frac{s-\frac{1}{2}-np_L}{\sigma_s}}^{\infty} (2\pi)^{-\frac{1}{2}} e^{-u^2/2} du \tag{2.10}$$

where the σ_s of (2.6) or (2.7) is evaluated with $p=p_L$ and $q=q_L$. We approximate σ_s as $\hat{V}p_L$, using (2.8). The lower limit of the right-hand integral in (2.10) becomes

$$\frac{s-\frac{1}{2}-np_L}{\hat{V}^{\frac{1}{2}}p_L^{\frac{1}{2}}}, \tag{2.11}$$

which must then equal u_α. Similarly,

$$\frac{s+\frac{1}{2}-np_U}{\hat{V}^{\frac{1}{2}}p_U^{\frac{1}{2}}} = u_{1-\alpha} = -u_\alpha. \tag{2.12}$$

Solving quadratic equations for p_L and p_U gives the normal approximation confidence limits for p,

$$p_{U0} = (2n^2)^{-1}[\hat{V}u_\alpha^2+(2s+1)n+\{[\hat{V}u_\alpha^2+(2s+1)n]^2 - (2s+1)^2n^2\}^{\frac{1}{2}}],$$

$$p_{L0} = (2n^2)^{-1}[\hat{V}u_\alpha^2+(2s-1)n-\{[\hat{V}u_\alpha^2+(2s-1)n]^2 - (2s-1)^2n^2\}^{\frac{1}{2}}],$$

$$0<p_{L0}<p_{U0}\leq 1. \qquad (2.13)$$

The ambiguity in sign is resolved by the knowledge that p_{U0} should exceed s/n, and s/n should exceed p_{L0}. If p_{U0} turns out to exceed 1, then p_U is taken as 1. If s=0, then $\hat{p}$ and $\hat{V}$ cannot be calculated; p_L=0, of course, and p_U is indeterminate, though we expect from (2.6) that $p_U \geq p_{UI}$ if $\lambda \geq p$, where p_{UI} is the upper confidence limit for independent trials, and in any case $p_U \leq 1-\alpha$, which is the limit for n=1.

Probably more accurate limits could be obtained by taking p as p_L or p_U throughout Var s, but a higher-degree equation would result and preclude a simple explicit solution. After the limits (2.13) have been calculated, they can be substituted for $\hat{p}$ in (2.8) and $\hat{V}$ corrected in (2.13) to produce corrected values of p_{U0} and p_{L0}. If these differ appreciably from the original values, a further iteration can be made, and so on. However, iteration seems unwarranted in view of the error of the normal approximation and the replacement of λ by $\hat{\lambda}$.

Since $n^{\frac{1}{2}}(\hat{\lambda}-\lambda,\hat{p}-p)$ has a limiting bivariate normal distribution with variance-covariance matrix

$$\begin{pmatrix} \lambda(1-\lambda)/p & q\lambda \\ q\lambda & pq(1-2p+\lambda)/(1-\lambda) \end{pmatrix} \qquad (2.14)$$

[Klotz, 1973, eq. (4.1)], it follows that, approximately, for large n,

$$P\left[\frac{p(\hat{\lambda}-\lambda)^2}{\lambda(1-\lambda)} - \frac{2(\hat{\lambda}-\lambda)(\hat{p}-p)}{1-2p+\lambda} + \frac{(1-\lambda)(\hat{p}-p)^2}{pq(1-2p+\lambda)} \leq \frac{\chi^2_{2,2\alpha}}{n}\right] = 1-2\alpha, \qquad (2.15)$$

where $\chi^2_{2,2\alpha}$ is the upper 200α percentage point of the chi-squared distribution with 2 degrees of freedom. The equality within the brackets of (2.15) defines the boundary of a confidence region for (λ,p). It is of degree 4 in p and 3 in λ.

An elliptical approximation for the region can be obtained by taking $p=\hat{p}$, $q=\hat{q}$, $\lambda=\hat{\lambda}$ in the variances and covariance. This may not be a good approximation for small p. In that case a better approximation is obtained by replacing only q by $\hat{q}$ (in its one occurrence) and p by $\hat{p}$ in the combination $1-2p+\lambda$. This reduces the degree in p to 2. Hence the boundary of the $1-2\alpha$ confidence region for (λ,p) for small p is given by the solutions of the quadratic equation in p,

$$p^2[\lambda(1-\lambda)^2 + 2\hat{q}\lambda(1-\lambda)(\hat{\lambda}-\lambda) + \hat{q}(1-2\hat{p}+\lambda)(\hat{\lambda}-\lambda)^2]$$
$$-p\lambda(1-\lambda)[2\hat{p}(1-\lambda) + 2\hat{p}\hat{q}(\hat{\lambda}-\lambda) + \chi^2_{2,2\alpha}\hat{q}(1-2\hat{p}+\lambda)/n]$$
$$+ \hat{p}^2\lambda(1-\lambda)^2 = 0, \qquad (2.16)$$

for assigned values of λ. An example of the application of (2.16) is given in Section 9 (and Figure 1 therein).

The probability that the bivariate confidence region includes the true (λ,p) is $1-2\alpha$ (aside from the normal approximation and replacement of p by $\hat{p}$ in three non-critical places). Hence the probability that the extreme values of p on the boundary of the region include the true p is at least $1-2\alpha$, whatever the value of λ. These extreme values provide a conservative confidence interval for p. If they enclose the interval of (2.13), they then furnish a check on the replacement of λ by $\hat{\lambda}$ in going from (2.7) to (2.8).

An approximate confidence interval for λ by itself can be obtained from (2.14). With probability $1-2\alpha$ in large samples,

$$\frac{|\hat{\lambda}-\lambda|}{[\lambda(1-\lambda)/(np)]^{\frac{1}{2}}} < u_\alpha.$$

Approximating np by $n\hat{p}=s$ and solving for λ gives the confidence limits

$$\frac{u_\alpha^2+2s\hat{\lambda}\pm[(u_\alpha^2+2s\hat{\lambda})^2-4s\hat{\lambda}^2(u_\alpha^2+s)]^{\frac{1}{2}}}{2(u_\alpha^2+s)} \qquad (2.17)$$

Rougher but simpler confidence limits for λ follow by replacing λ by $\hat{\lambda}$ in the standard deviation:

$$\hat{\lambda} \pm u_\alpha[\hat{\lambda}(1-\hat{\lambda})/s]^{\frac{1}{2}}. \qquad (2.18)$$

Examples of (2.13) and (2.18) are given in Sections 9 and 12.

If s is very small or very close to n, the normal approximation of the probability sums in (2.5) is very poor. However, in those cases, at least for s=0, 1, n-1, or n, it is practical to calculate the exact probabilities; see Section 7.

3. CONFIDENCE INTERVALS FROM THE EDGEWORTH TWO-TERM APPROXIMATION

Better approximations for confidence limits for p should be obtainable from more accurate representations of the sums in (2.5) by Johnson transformations, Pearson curves, or Edgeworth expansions. The Edgeworth two-term approximation for the probability density function of a standardized variable (mean 0 and variance 1) is [Cramér, 1946, eq. (17.7.3)]

$$f(u) = \phi(u) - \frac{\gamma_1}{3!}\phi^{(3)}(u) \qquad (3.1)$$

where γ_1 is the standardized third central moment,

$$\phi(u) = (2\pi)^{-1/2}e^{-u^2/2},$$

and $\phi^{(3)}(u)$ is the third derivative of $\phi(u)$. The corresponding cumulative distribution function is

$$\int_{-\infty}^{u} f(x)dx = \Phi(u) - \frac{\gamma_1}{3!}(u^2-1)\phi(u) \qquad (3.2)$$

where

$$\Phi(u) = \int_{-\infty}^{u} \phi(x)dx.$$

Thus, if γ_1 is known, the Edgeworth two-term approximation for discrete probability sums such as in (2.5) can be evaluated for any given u from readily available tables (National Bureau of Standards, 1953; Abramowitz and Stegun, 1964).

The third central moment of s, derived by Gabriel (1959), is

$$E[(s-Es)^3] = npq(q-p) + \frac{6pq(q-p)\rho}{(1-\rho)^3}[(n-1)(1-\rho^{n+1}) - (n+1)(\rho-\rho^n)], \quad |\rho|<1. \qquad (3.3)$$

For large n and ρ bounded from 1, the powers of ρ are negligible in (3.3).

Approximate confidence limits for p can therefore be obtained from the two-term Edgeworth expansion by substituting (3.2) for the left-hand sums in (2.5) with u taken as (2.11) and the left-hand member of (2.12) respectively. Since the term in γ_1 approaches zero as n becomes infinite, it can be approximated by substituting sample values for p, q, and ρ, or, better still, p_{L0} and p_{U0} from (2.13) since (2.5) is an equation to be solved for p. The solution can then be iterated to eliminate any approximation except the use of $\hat{\lambda}$ for λ and the use of the Edgeworth expansion for the sums in (2.5).

To be explicit, we let

$$c_{Li} = (\hat{\lambda}-p_{Li})/q_{Li}, \qquad (|c_{Li}|<1),$$

$$V_{Li} = \frac{nq_{Li}}{(1-c_{Li})^2}(1-\rho_{Li}^2-2\rho_{Li}/n),$$

$$B_{Li} = (6V_{Li}^{3/2})^{-1} q_{Li}(1-2p_{Li})\{n+6c_{Li}(1-c_{Li})^{-3}[n-1-(n+1)\rho_{Li}]\}, \qquad i = 0,1,2,.... \qquad (3.4)$$

[For $\hat{\lambda}$ very near 1 and n small, the term $-2\rho_{Li}/n$ of V_{Li} should be multiplied by $(1-\rho_{Li}^n)$ in accordance with (2.6); likewise the last bracket in (3.4) should be replaced by the last bracket in (3.3) with $\rho=\rho_{Li}$.] The notation for upper limit quantities is analogous, with L replaced by U throughout. If in fact p is as small as p_{Li}, then the standardized third central moment of S is

$$\gamma_1 = \frac{E[(s-Es)^3]}{\sigma_s^3} \doteq 6B_{Li}p_{Li}^{-\frac{1}{2}} \doteq 6B_{L,i-1}p_{Li}^{-\frac{1}{2}}. \qquad (3.5)$$

It follows from (3.2) - (3.4) that the first equation of (2.5) is approximated by

$$\Phi\left(\frac{s-\frac{1}{2}-np_{Li}}{V_{L,i-1}^{\frac{1}{2}}p_{Li}^{\frac{1}{2}}}\right) - B_{L,i-1}p_{Li}^{-\frac{1}{2}}\left[\left(\frac{s-\frac{1}{2}-np_{Li}}{V_{L,i-1}^{\frac{1}{2}}p_{Li}^{\frac{1}{2}}}\right)^2 - 1\right]\phi\left(\frac{s-\frac{1}{2}-np_{Li}}{V_{L,i-1}^{\frac{1}{2}}p_{Li}^{\frac{1}{2}}}\right) = 1-\alpha,$$

$$i = 1,2,\ldots, \qquad (3.6)$$

which is to be solved for p_{Li}. Since the second term on the left-hand side is expected to be relatively small, an iterative solution can be obtained by replacing p_{Li} by $p_{L,i-1}$ in it. Hence we define

$$\alpha_{Li} = \alpha - B_{L,i-1}p_{L,i-1}^{-\frac{1}{2}}(u_{\alpha_{L,i-1}}^2 - 1)\phi(u_{\alpha_{L,i-1}}) \qquad (3.7)$$

where $i=1,2,\ldots$; $\alpha_{L0}=\alpha$; and $0<\alpha_{Li}<\frac{1}{2}$. (If $\alpha_{Li}\leq 0$ or $\alpha_{Li}\geq\frac{1}{2}$, we stop the iteration and set $p_L=0$.) Thence (3.5) is replaced by

$$\Phi\left(\frac{s-\frac{1}{2}-np_{Li}}{V_{L,i-1}^{\frac{1}{2}}p_{Li}^{\frac{1}{2}}}\right) = 1 - \alpha_i \qquad (3.8)$$

or

$$\frac{s-\frac{1}{2}-np_{Li}}{V_{L,i-1}^{\frac{1}{2}}p_{Li}^{\frac{1}{2}}} = u_{\alpha_{Li}}, \qquad (3.9)$$

the left-hand member of which is of the same form as (2.11). The solution is thus of the same form as (2.13). The upper limit p_{Ui} follows analogously, except that

$$\alpha_{Ui} = \alpha + B_{U,i-1}p_{U,i-1}^{-\frac{1}{2}}(u_{\alpha_{U,i-1}}^2 - 1)\phi(u_{\alpha_{U,i-1}}) \qquad (3.10)$$

where $i=1,2,\ldots$; $\alpha_{U0}=\alpha$; and $0<\alpha_{Ui}<\frac{1}{2}$. (If $\alpha_{Ui}\leq 0$ or $\alpha_{Ui}>\frac{1}{2}$, we stop and set $p_U=1$). Thus

$$p_{Ui} = (2n^2)^{-1}\{V_{U,i-1}u_{\alpha_{Ui}}^2 + (2s+1)n + [\{V_{U,i-1}u_{\alpha_{Ui}}^2 + (2s+1)n\}^2 - (2s+1)^2n^2]^{\frac{1}{2}}\}$$

$$p_{Li} = (2n^2)^{-1}\{V_{L,i-1}u_{\alpha_{Li}}^2 + (2s-1)n - [\{V_{L,i-1}u_{\alpha_{Li}}^2 + (2s-1)n\}^2 - (2s-1)^2n^2]^{\frac{1}{2}}\}$$

$$0 \le p_{Li} < p_{Ui} \le 1, \quad i=1,2,.... \quad (3.11)$$

The solutions p_U and p_L will have been obtained when successive iterates agree to an acceptable number of significant figures; two or three seem sufficient in practice. Examples of application of (3.11) and (2.13) are given in Section 9.

4. CONFIDENCE INTERVALS FROM THE EDGEWORTH FOUR-TERM APPROXIMATION

If the first four moments of a random variable are available, its cumulative distribution function can be further approximated by the Edgeworth expansion [Cramér, 1946, eq. (17.7.3)]

$$\int_{-\infty}^{u} f(x)dx = \Phi(u) - \frac{\gamma_1}{3!}\phi^{(2)}(u) + \frac{\gamma_2}{4!}\phi^{(3)}(u) + \frac{10\gamma_1^2}{6!}\phi^{(5)}(u) \quad (4.1)$$

where u has mean 0 and variance 1 and γ_2 is the kurtosis measure (standardized fourth central moment less 3). Substitution for the sums in (2.5) may then produce an iterative solution for a higher-order approximation for the confidence limits for p.

Gabriel (1959) gives the asymptotic value for the γ_2 value of s:

$$\gamma_2 \sim \frac{1-6pq}{npq} \cdot \frac{1+10\rho+\rho^2}{1-\rho^2} . \quad (4.2)$$

To obtain an iterative solution for p_L, we approximate $p\gamma_2/4!$ by

$$C'_{Li} = \frac{1-6p'_{Li}q'_{Li}}{24nq'_{Li}} \cdot \frac{1+10\rho'_{Li}+\rho'^2_{Li}}{1-\rho'^2_{Li}}, \quad i=0,1,2,..., \quad (4.3)$$

where the primes are used to distinguish four-term iterates from two-term iterates. We define C'_{Ui} analogously. Substitution of (4.1) for the sums in (2.5) leads to iterative solutions of the precise form (3.11) with α'_{Ui} and α'_{Li} replacing α_{Ui} and α_{Li}, where

$$\alpha'_{Ui} = \alpha + B'_{U,i-1}p'^{-\frac{1}{2}}_{U,i-1}(u^2_{\alpha'_{U,i-1}} - 1)\phi(u_{\alpha'_{U,i-1}})$$
$$- p'^{-1}_{U,i-1}[C'_{U,i-1}\phi^{(3)}(u_{\alpha'_{U,i-1}}) + \tfrac{1}{2}B'^{2}_{U,i-1}\phi^{(5)}(u_{\alpha'_{U,i-1}})],$$

$$\alpha'_{Li} = \alpha - B'_{L,i-1}p'^{-\frac{1}{2}}_{L,i-1}(u^2_{\alpha'_{L,i-1}} - 1)\phi(u_{\alpha'_{L,i-1}}) \qquad (4.4)$$
$$+ p'^{-1}_{L,i-1}[C'_{L,i-1}\phi^{(3)}(u_{\alpha'_{L,i-1}}) + \tfrac{1}{2}B'^{2}_{L,i-1}\phi^{(5)}(u_{\alpha'_{L,i-1}})],$$

where $i=1,2,\ldots$; $\alpha'_{U0}=\alpha'_{L0}=\alpha$; $0<\alpha'_{Ui}<\frac{1}{2}$; $0<\alpha'_{Li}<\frac{1}{2}$. (If $\alpha'_{Ui}\leq 0$ or $\alpha'_{Ui}\geq\frac{1}{2}$, we stop and set $p_U=1$; if $\alpha'_{Li}\leq 0$ or $\alpha'_{Li}\geq\frac{1}{2}$, we stop and set $p_L=0$.) The quantities $B'_{U,i-1}$ and $B'_{L,i-1}$ are given by (3.4) with primes added, and $C'_{U,i-1}$ and $C'_{L,i-1}$ by (4.3), U being substituted for L for the upper limit quantities. The derivatives $\phi^{(j)}(x)$ are tabulated by Abramowitz and Stegun (1964, Table 26.1).

The effect of using three-moment iterates at various points may be negligible. If the Edgeworth two-term approximation has been carried to convergence prior to starting the four-term approximation, then it should be a better starting value than the normal approximation.

5. CONFIDENCE INTERVALS FROM THE PEARSON SYSTEM APPROXIMATION

When the third and fourth moments of s are approximated as in Section 4, the distribution of s can alternatively be represented by the Pearson system of distributions. By virtue of tabulation of the percentage points of the Pearson distributions as functions of $\beta_1=\gamma_1^2$ and $\beta_2=\gamma_2+3$ by E.S. Pearson and

M. Merrington (1951) and by Johnson et al. (1963) (reproduced in Pearson and Hartley, 1966, 1972), an iterative solution for confidence limits for p can be found with less calculation than required by the Edgeworth four-term approximation.

Using the notation of Sections 3 and 4, we calculate

$$\beta_{1L,i-1} = 36B_{L,i-1}^2 p_{L,i-1}^{-1},$$

$$\beta_{2L,i-1} = 3 + 24C_{L,i-1} p_{L,i-1}^{-1}, \quad i=1,2,\ldots, \qquad (5.1)$$

where B_{Li} and C_{Li} are given by (3.4) and (4.3) [deleting the primes in (4.3)]. We then read $u_{\alpha_{Li}}$ as the upper 100α percent point of Table 32 in Pearson and Hartley (1972) and substitute it in (3.11) to find p_{Li}. (This assumes $p_{L,i-1} \leq 0.5$.) We can then substitute in (5.1) to iterate if necessary. Similarly, we find p_U starting with (5.1) with L replaced by U and $u_{\alpha_{Ui}}$ found as the lower 100α percent point of Pearson and Hartley's Table 32. (If $p_{U,i-1} > 0.5$, substitute "upper".)

The derivation of these approximate limits follows immediately from the definition of β_1 and β_2 and the approximation of the sums in (2.5) by probability integrals of Pearson curves.

All of the approximations in Sections 2 through 5 would be expected to be better the larger the sample size n is, the smaller the confidence level $1-2\alpha$ is, the closer the conditional probability λ is to the unconditional probability p, and the closer p or $\hat{p}$ is to 1/2. If $\hat{p}$ is near zero, then p_L is even smaller, and the occurrence of $p_{L,i-1}$ in the denominator of (3.7) and of $p'_{L,i-1}$ in the denominator of (4.4) may result in values of α_{Li} and α'_{Li} outside the range (0,0.5). Thus for $\hat{p}$ near zero, p_L may be relatively poorly approximated, if at all, while p_U is well approximated. However, $p_L=0$ is probably a good approximation anyway, and p_L is often not of interest for small $\hat{p}$, an upper limit only being desired. Similarly, the Pearson system approximation may fall short by yielding values of β_1 or β_2 beyond the limits of Table 42 of Pearson and Hartley's Vol. I (1966) or Table 32 of their Vol. II (1972) (less often

for the latter), but it may be sufficient then to use the tabular value for the nearest (β_1,β_2). (Some extrapolation of the tables is feasible also.) The trial values $\hat{\rho}$ in (2.8) and ρ_{Li} in (3.4) (and ρ_{Ui} likewise) may exceed 1 in absolute value; the approximations then fail. The solutions tend to fail if the trial values of ρ fall close to +1 even though below it. Just how widely p, λ, n, and α may vary is investigated empirically in Section 11.

When s=0, $\hat{\lambda}$ and λ^* cannot be calculated. Consequently none of the confidence intervals of Sections 2-6 can be calculated. However, if an upper bound to λ (less than 1) can be established from prior experience, this bound can be substituted for $\hat{\lambda}$ in the limit formulas to obtain conservative (overly long) confidence intervals for p. The interval based on independent trials (Crow, 1974) is too short, so the "true" interval is somewhere between these two.

6. MODIFIED POISSON-ANDERSON-BURSTEIN APPROXIMATE CONFIDENCE INTERVALS

The approximating probability distributions discussed thus far may tend to be better approximations of the distribution of S the closer p is to ½. Is there an approximation that, like the Poisson relative to the binomial, tends to be better the smaller p is? Any answer is complicated by the need to estimate λ, for which a large value of s is desirable. The answer provided here is a rough but simple modification of Anderson and Burstein's improvement (1967, 1968; Crow, 1974) of the Poisson approximation of binomial confidence limits.

The roughest sort of confidence limits are those obtained as the point estimate of the parameter plus and minus a factor times the estimated standard error of the point estimate, based on asymptotic normality. Such symmetric limits become

unsatisfactory for asymmetric distributions, for example when p is very small. An alternative is to achieve asymmetry by lengthening the confidence interval of Anderson and Burstein by the asymptotic factor in the standard error of s resulting from dependence. Thus we get from (2.7), for $\hat{p}<0.1$,

$$p_U = \hat{p} + (p_{UI}-\hat{p})\left(\frac{1+\hat{\rho}}{1-\hat{\rho}}\right)^{1/2},$$

$$p_L = \hat{p} - (\hat{p}-p_{LI})\left(\frac{1+\hat{\rho}}{1-\hat{\rho}}\right)^{1/2}, \qquad (\text{if } p_L \geq 0), \qquad (6.1)$$

where p_{UI} and p_{LI} are the Anderson-Burstein limits for independent trials (Crow, 1974) and $\hat{\rho}=(\hat{\lambda}-\hat{p})/\hat{q}$. If the p_L calculated from (6.1) turns out to be negative, then (6.1) should be replaced by

$$p_U = (p_{UI}-p_{LI})\left(\frac{1+\hat{\rho}}{1-\hat{\rho}}\right)^{1/2}, \qquad p_L = 0. \qquad (6.2)$$

Despite their limitations, (6.1) and (6.2) may be better than the normal, Edgeworth, or Pearson approximation limits in certain regions of the (p,λ,n,α) space; see Section 11.

For small $\hat{p}$ it may not be of interest to bound p below (other than by zero). An upper $100(1-2\alpha)$ percent confidence limit for p may then be obtained that is smaller than the upper end of the central $100(1-2\alpha)$ percent confidence interval. It is in fact simply the upper end of the central $100(1-4\alpha)$ percent interval.

7. COMPARISON WITH EXACT CONFIDENCE LIMITS

Before considering the exact limits in general, we shall provide them for the cases of zero or one error, s=0 or s=1, for then none of the approximations except perhaps the modified Poisson-Anderson-Burstein is satisfactory and it is practical to use the exact limits. However, a prior value of λ must be available if s=0, as the sample provides no information on λ.

From (1.1)-(1.6) and (2.5),

$$P[s=0|p,\lambda,n] = f(0|p,\lambda,n)$$
$$= q[(1-2p+\lambda p)/q]^{n-1}, \qquad (7.1)$$

$$P[s=1|p,\lambda,n] = f(1|p,\lambda,n)$$
$$= \sum_{i=1}^{n} P[1 \text{ on } i\text{th trial } \& \ 0 \text{ on others}]$$
$$= p(1-\lambda)[(1-2p+\lambda p)/q]^{n-2}$$
$$+ \sum_{i=2}^{n-1} q(1-\lambda)(p/q)(1-\lambda)[((1-2p+\lambda p)/q]^{n-3}$$
$$+ q[(1-2p+\lambda p)/q]^{n-2}(1-\lambda)p/q$$
$$= 2p(1-\lambda)[(1-2p+\lambda p)/q]^{n-2}$$
$$+ (n-2)p(1-\lambda)^2[(1-2p+\lambda p)/q]^{n-3}$$
$$= (p/q)(1-\lambda)[nq(1-\lambda)+2(\lambda-p)]$$
$$\cdot [(1-2p+\lambda p)/q]^{n-3}. \qquad (7.2)$$

If s=0, it is intuitively evident that there is no information on λ in the sample. This is confirmed by the fact that the probability or likelihood of the sample s=0 is 1 for $p=\hat{p}=0$ whatever the value of λ is. Thus we cannot calculate p_U from (2.5) and (7.1) with any $\hat{\lambda}$ or λ^* from the sample, and we cannot calculate p_U at all unless we are furnished with a prior value of λ or an upper bound on λ. Since $\hat{p}=0$, we take $p_L=0$ also.

Provided with a prior value of λ or an upper bound on λ, we can solve the right-hand equation (2.5), with (7.1) substituted, by iteration. From

$$q_U[1-(1-\lambda)p_U/q_U]^{n-1} = \alpha$$

we find that

$$p_U = \frac{1-(\alpha/q_U)^{1/(n-1)}}{2-\lambda-(\alpha/q_U)^{1/(n-1)}} \cdot \qquad (7.3)$$

Example 1. $\lambda=.3$, $\alpha=.05$, $n=50$, $s=0$. An initial approximation to substitute for q_U on the right-hand side of (7.3) can be found from the modified Poisson-Anderson-Burstein approximation (6.2). From table 1 of OT Report 74-51 (Crow, 1974), $U=3.00$, so

$$q_U \doteq q_0 = 1-p_0 = 1-\frac{3.00}{51.5}\left(\frac{1.3}{.7}\right)^{1/2} = 1-.079 \doteq .92$$

$$p_1 = \frac{1-(.05/.92)^{1/49}}{1.7-(.05/.92)^{1/49}} = .07616,$$

$$p_2 = .076253,\ p_3 = .076252 = p_U,$$

accurate to at least 5 decimal places (DP). If $\lambda=.8$ rather than .3, then we get $p_3=.21464=p_U$ to at least 3 DP.

Since p_U is a monotone increasing function of λ, a prior upper bound on λ would enable us to compute from (7.3) a conservative upper confidence limit on p. If .8 were an upper bound on λ in Example 1, we could say that we are 90% confident that p is .215 or less. (Aside from the approximations, the confidence limits of sections 2-6 are conservative anyway because the discreteness of s precludes attaining the exact confidence level $1-2\alpha$, short of artificial post-sampling randomization, and we always err on the upper side.) If we are interested at the outset (prior to seeing the data) only in upper limits, then we could say that we are 95% confident that p is .215 or less. As λ approaches 1, we have effectively in the limit just one independent observation and p_U approaches $1-\alpha$ if $s=0$. In Example 1, $p_U \to .95$ as $\lambda \to 1$. With $\lambda=.999$, $p_U=.9150$. With independence, $\lambda=p$, and $p_U=1-\alpha^{1/n}=.0582$. If $\lambda=0$ (extreme negative dependence), $p_U=.0550$. Thus, in the example $n=50$, $s=0$, $1-\alpha=.95$, as λ varies over its entire range from 0 to 1, p_U increases monotonically from .0550 to .9500.

Proceeding similarly for s=1, we have from (2.5), (7.1), and (7.2) after some algebra

$$p_U = \frac{1-A}{2-\lambda-A}, \tag{7.4}$$

where

$$A = \left\{ \frac{\alpha q_U}{1+p_U[(n-2)(1-\lambda)^2-2] - p_U^2[(n-1)(1-\lambda)^2-1]} \right\}^{1/(n-3)}.$$

Like (7.3), this can be solved iteratively using the Anderson-Burstein approximation initially.

Example 2. $\lambda=.3$, $\alpha=.05$, $n=50$, $s=1$. Then $\hat{p}=.025$, $p_L=0$ by the Anderson-Burstein approximation, and

$$p_0 = \frac{4.7}{51.85}\left(\frac{1.3}{.7}\right)^{1/2} = .124, \quad A_0 = .91201,$$

$$p_1 = .1116, \quad p_2 = .1097, \quad p_3 = .10967 = p_U,$$

correct to at least 4 DP. If $\lambda=.8$ rather than .3 then we get $p_3=.24782=p_U$ to at least 4 DP. If $\lambda=p$, then $p_3=.0914=p_U$ to 4 DP. If $\lambda=0$, then $p_3=.0854=p_U$ to 4 DP. As λ varies from 0 to 1, p_U varies from .0854 to 1.

If $s=1$, it is possible to calculate $\hat{\lambda}$ and λ^* from the sample using (2.2) and (2.4), but the only possible value of r is 0, and the only possible values of t are 0 or 1. With either value of t, the value of $\hat{\lambda}$ in (2.2) [as well as of λ^* in (2.4)] is seen to be 0. The estimate (2.2) for λ is not necessarily good for small s (cf. Klotz, 1973, Sec. 4), and this is definitely the case for $s=1$; to take $\hat{\lambda}=0$ always when λ may in general vary from $\max(0,2-p^{-1})$ up to 1 is unrealistic. It is to be hoped that some prior value or upper bound for λ can be assumed.

For $s\geq 2$, we use Ladd's (1973,1975) algorithm for sequentially calculating the cumulative probabilities (2.5). The algorithm follows all the possible "random walks" from the first trial through the second,...,(K-1)th, and Kth trials (transmissions in the telecommunications application) with a fork in the path to success or failure (error or correct transmission, hit or miss)

at each trial. We present the formulas in Ladd's notation except for using our p and q instead of his P and 1-P and $f_{in}=f(i|p,\lambda,n)$ for the probability of i successes in n trials, as in (2.5), instead of his $\Phi_{j,k}(P,P_{hh},P_{hm})$. Let

$$\begin{aligned} P_{hh} &= P[X_i=1|X_{i-1}=1] = \lambda, \\ 1-P_{hh} = P_{mh} &= P[X_i=0|X_{i-1}=1], \\ P_{hm} &= P[X_i=1|X_{i-1}=0], \\ 1-P_{hm} = P_{mm} &= P[X_i=0|X_{i-1}=0] \end{aligned}$$

[cf. (1.2)-(1.6)]. Then

$$f_{i,n} = \xi_{i,n} + \eta_{i,n} \tag{7.5}$$

where

$$\begin{aligned} \xi_{i,n} &= P[\text{i successes in n trials \& nth trial succeeds}], \\ \eta_{i,n} &= P[\text{i successes in n trials \& nth trial fails}]; \end{aligned}$$

$$\begin{aligned} \xi_{0,1} &= 0, \quad \xi_{1,1} = p, \quad f_{1,1} = p, \\ \eta_{0,1} &= q, \quad \eta_{1,1} = 0, \quad f_{0,1} = q; \end{aligned}$$

$$\begin{aligned} \xi_{0,n} &= 0, \\ \eta_{0,n} &= \eta_{0,n-1}P_{mm} = qP_{mm}^{n-1} = f_{0,n}; \end{aligned}$$

$$\begin{aligned} \xi_{i,n} &= \xi_{i-1,n-1}P_{hh} + \eta_{i-1,n-1}P_{hm}, \\ \eta_{i,n} &= \xi_{i,n-1}P_{mh} + \eta_{i,n-1}P_{mm}, \quad i=1,2,\ldots n-1; \end{aligned}$$

$$\begin{aligned} \xi_{n,n} &= \xi_{n-1,n-1}P_{hh} = pP_{hh}^{n-1} = f_{n,n}, \\ \eta_{n,n} &= 0 \qquad\qquad n=1,2,\ldots. \end{aligned} \tag{7.6}$$

The cumulative probabilities (2.5) are then found by summing (7.5).

A computer program was written to solve (2.5) for p_L and p_U for λ, s, and n using the exact probabilities (7.5). It can

of course be used as an approximation when λ is unknown by substituting $\hat{\lambda}$ from (2.2). For large n it is more economical and sufficiently accurate to use the program in Section 8 for the approximate intervals of Sections 2-6, which can be applied with a prior value of λ rather than $\hat{\lambda}$ also. We proceed to illustrate the application of both programs.

Example 3. λ=.3, α=.05, n=50, s=5. Then $\hat{p}$=.1. The "exact" 90% confidence limits from the computer program based on (7.6) are p_U=.2105 and p_L=.02880. The Anderson-Burstein limits (6.1)-(6.2) using $\hat{\rho}=(\lambda-\hat{p})/\hat{q}$=.222 are

$$p_U = .1 + (\frac{10.5}{52.75} - .1)(\frac{11}{7})^{1/2} = .224,$$

$$p_L = .1 - (.1-\frac{1.97}{48.98})(\frac{11}{7})^{1/2} = .025.$$

The approximate limits from the other computer program are

	Normal	Edgeworth 2	Edgeworth 4	Pearson
p_U	.24546	.21030	.21060	.21031
p_L	.042646	.030286	.036049	.034111

The last three p_U approximations are thus excellent, the other two p_U approximations are a bit high. All the p_L approximations except the Anderson-Burstein are somewhat high. The times for running the "exact" and "approximate" programs on the CDC 6600 computer were 3.48 and 1.67 seconds, respectively. The normal and Anderson-Burstein formulas can of course be evaluated quickly on a pocket calculator.

Example 4. (a) Same as Example 3 except $\hat{\lambda}$ calculated from r=0, s=5, t=0. From (2.2), $\hat{\lambda}$=0. Since the asymptotic standard deviation of $\hat{\lambda}$ for λ=.3 and s=5 is approximately $(.3x.7/5)^{1/2}$=.2, the value $\hat{\lambda}$=0 is a not improbable sample result when λ=.3. The Anderson-Burstein limits (6.1)-(6.2) are, since $\hat{\rho}=(\hat{\lambda}-\hat{p})/\hat{q}$=-1/9, p_U=.189, p_L=.047. The "exact" limits from the computer program

under the assumption $\lambda=\hat{\lambda}=0$ are $p_U=.1805$, $p_L=.04201$. The approximate limits from the other computer program are

	Normal	Edgeworth 2	Edgeworth 4	Pearson
p_U	.19803	.18113	.18113	.18136
p_L	.04708	.04078	.041668	.041175

All of the approximations except the normal seem close enough in practice. The times for running the "exact" and "approximate" programs were 2.99 and 1.54 seconds, respectively.

Example 4. (b) Same as Example 4(a) except $\hat{\lambda}$ calculated from r=3, s=5, t=0. From (2.2) $\hat{\lambda}=.597$. This also is a not improbable sample result when $\lambda=.3$. The Anderson-Burstein limits (6.2) are $p_U=.296$, $p_L=0$. The "exact" limits from the computer program under the assumption $\lambda=\hat{\lambda}=.597$ are $p_U=.2641$, $p_L=0$. The approximate limits from the other computer program are

	Normal	Edgeworth 2	Edgeworth 4	Pearson
p_U	.34851	.26638	.26695	.26784
p_L	.025493	0	.02687	.019904

The normal is again the poorest approximation, again on the high side. Only the Anderson-Burstein and Edgeworth 2 values of p_L are as close as desirable. The changes in p_U and p_L with possible variations of $\hat{\lambda}$ in Examples 3 and 4 are interesting and perhaps not disturbing for this small sample size. The times for running the "exact" and "approximate" programs were 2.01 and 1.57 seconds, respectively.

Example 5. $\lambda=.3$, $\alpha=.05$, n=150, s=15. Then $\hat{p}=.1$ as in Examples 3-4, but the sample size is 3 times as large. That is the largest sample size that can be run on the CDC 6600 with the "exact" program as written. The "exact" 90% limits from the computer program are $p_U=.1590$ and $p_L=.05377$. The Anderson-Burstein limits (6.1) taking $\hat{\lambda}=\lambda$ are $p_U=.163$ and $p_L=.053$. The approximate limits from the other computer program are

	Normal	Edgeworth 2	Edgeworth 4	Pearson
p_U	.16886	.15937	.15959	.15932
p_L	.058197	.052996	.053823	.053629.

Here the approximations, except for the Anderson-Burstein, are substantially better than in Examples 3 and 4, because of the increase in sample size. The times for running the "exact" and "approximate" programs on the CDC 6600 computer were 24.94 and 1.68 seconds, respectively. Thus the exact program is the faster only for sample sizes up to the order of 100 and is impractical for the sample sizes needed in telecommunications.

Example 6. (a) Same as Example 5 except $\hat{\lambda}$ calculated from r=2, s=15, t=0. From (2.2), $\hat{\lambda}$=.133. Since the asymptotic standard deviation of $\hat{\lambda}$ for λ=.3 and s=15 is approximately $(.3x.7/15)^{1/2}$=.118, the value r=2 is not an improbable sample result when λ=.3. The Anderson-Burstein limits (6.1) are p_U=.152, p_L=.061. The "exact" limits from the computer program under the assumption $\lambda=\hat{\lambda}$=.133 are p_U=.14882, p_L=.060204. The approximate limits from the other program are

	Normal	Edgeworth 2	Edgeworth 4	Pearson
p_U	.15539	.14908	.14916	.14919
p_L	.06341	.059816	.060145	.060116

The times for running the "exact" and "approximate" programs were 28.80 and 1.63 seconds, respectively.

Example 6. (b) Same as Example 6(a) except $\hat{\lambda}$ calculated from r=8, s=15, t=0. From (2.2), $\hat{\lambda}$=.532. This also is a not improbable sample result when λ=.3. The Anderson-Burstein limits (6.1) are p_U=.184, and p_L=.036. The exact limits from the computer program under the assumption $\lambda=\hat{\lambda}$=0 are p_U=.18014, p_L=.042554. The approximate limits from the other program are

	Normal	Edgeworth 2	Edgeworth 4	Pearson
p_U	.19878	.18095	.18146	.18036
p_L	.049188	.040640	.043013	.042284.

As in Examples 4(a) and 4(b), the confidence intervals for large $\hat{\lambda}$ are longer than for small $\hat{\lambda}$, as expected, but the variation is probably acceptable, and the number of intervals that are "too short" (i.e., fail to cover the true p frequently enough) will in the long run be essentially balanced by those that are "too long" (i.e., cover the true p too frequently). The times for running the "exact" and "approximate" programs were 25.72 and 1.67 seconds, respectively.

The results of Examples 3-6 are summarized in Tables 1 and 2 to three decimal places. Since all of the iterative solutions (Edgeworth 2- and 4-term and Pearson) agree to 3 DP usually, only the average values are given for these, under a common heading. The following conclusions can be drawn, although they cannot necessarily be generalized to other confidence levels or sample characteristics:

1. The Edgeworth 2-term and 4-term and Pearson limits agree to 2 or 3 DP with the "exact" limits as well as with each other.
2. The Poisson-Anderson-Burstein lower limit is fairly close to the exact lower limit, either above or below, while the upper limit is higher.
3. The normal limits are the poorest approximations to the exact limits, both lower and upper limits being too large. Thus the length of the interval does not differ as much relatively as the limits.
4. The change in either exact or approximate limits due to variation in $\hat{\lambda}$ (and thus in the number r of pairs of successive errors) is much greater than the difference between any of the types of limits. This

Table 1. Summary of Examples 3-6 Comparing Exact and Approximate 90% Confidence Limits for p When $\hat{p}=.1$, $\lambda=.3$, and Either n=50 or n=150

Ex.	$\hat{\lambda}$	Exact		And.-Burstein		Normal		Edgeworth 2, 4 & Pearson	
		p_L	p_U	p_L	p_U	p_L	p_U	p_L	p_U
	n=50								
4a	low	.042	.180	.047	.189	.047	.198	.041	.181
3	λ	.029	.210	.025	.224	.043	.245	.033	.210
4b	high	.000	.264	.000	.296	.025	.349	.02	.267
	n=150								
6a	low	.060	.149	.061	.152	.063	.155	.060	.149
5	λ	.054	.159	.053	.163	.058	.169	.054	.159
6b	high	.043	.180	.036	.184	.049	.199	.042	.181

Table 2. Computer Execution Times Required for Table 1 Results* (90% Confidence Limits for p When $\hat{p}=.1$ and $\lambda=.3$)

n	Example	$\hat{\lambda}$	Exact Limits	Approximate Limits
50	4a	low	2.99 sec	1.54 sec
	3	λ	3.48	1.67
	4b	high	2.01	1.57
150	6a	low	28.80	1.63
	5	λ	24.94	1.68
	6b	high	25.72	1.67

*The above times are in addition to the constant compilation time, which is 1.81 sec for the exact limits program and about 7 sec for the approximate limits program. Both compilation and execution times will vary from computer to computer.

does not mean that the limits are wrong due to ignorance of λ, for the intervals still cover the true value p in approximately 90% of the cases, as verified in Section 11, and that is all that they are supposed to do. It does mean that the approximations are quite accurate enough. (The limits for the low value of $\hat{\lambda}$ are about the same as those obtained assuming independence.)

5. The change in limits due to variation in $\hat{\lambda}$ decreases substantially with sample size. For example, the ratio of confidence interval lengths at high and low $\hat{\lambda}$ is about 2.0 for n=50 and about 1.6 for n=150.
6. The computation time for the exact limits increases rapidly with sample size, whereas that for the approximate limits stays essentially constant. For the case considered, the exact limits take less time for sample sizes less than about 100, more time beyond that point.

8. COMPUTER PROGRAM FOR APPROXIMATE CONFIDENCE LIMITS

This section outlines the important features of the main program (CONLIM) and each of its subroutines for calculating the approximate confidence limits for p and λ derived in Sections 2-6. Comments are scattered throughout the listing (Appendix B) to further aid the user in understanding the logic of the program.

The program is written in FORTRAN IV for the CDC 6600. The dimensioned variables occupy 13,600 words. The compilation time is about 7 seconds, and the execution time is about 2 seconds.

The program utilizes three routines that are not available to all users. The upper 100α percentage point of the standardized normal distribution, u_α (eq. 2.9), and the upper 200α percentage point of the chi-squared distribution with 2 degrees of freedom,

$\chi^2_{2,2\alpha}$ (eq. 2.15), are both determined by routines from the International Mathematical Statistical Libraries, Inc. Also the confidence region for (λ,p) is plotted (subroutine PLOT) on the CDC-250 microfilm recorder using routines developed by NOAA.

Program CONLIM. This is the main calling program. It provides for the five basic input variables n, s, r, t, and α (defined in section 2) and the four precision variables:

1. NPTS = number of points used to define the boundary of the confidence region for (λ,p),
2. MAXIT = maximum number of iterations tolerated for the Edgeworth and Pearson system approximations,
3. MINSIG = minimum number of significant figures of agreement desired from successive iterations (relative precision),
4. MINDEC = minimum number of decimal places of agreement desired from successive iterations (absolute precision).

The program calls subroutine DEPEND which, in turn, calls 8 other subroutines. The program also provides printed output of the confidence limits for λ and p and the points of tangency for the confidence region for (λ,p). See Figure 1 which shows the logic among the subroutines.

Subroutine DEPEND. This subroutine calls 8 subroutines (PARAM, REGION, PLOT, LIMLAM, NORMAL, EDGEW, PSA, and ANDBUR). The first, PARAM, defines some basic parameters, and must be called first. After that, the calling order is immaterial except that REGION must precede PLOT and NORMAL must precede both EDGEW and PSA.

Subroutine PARAM. This subroutine calculates $\hat{p}$, $\hat{\lambda}$, λ^*, $\tilde{\lambda}$, and $\hat{\rho}$, which are defined in Section 2 and Section 11.3.

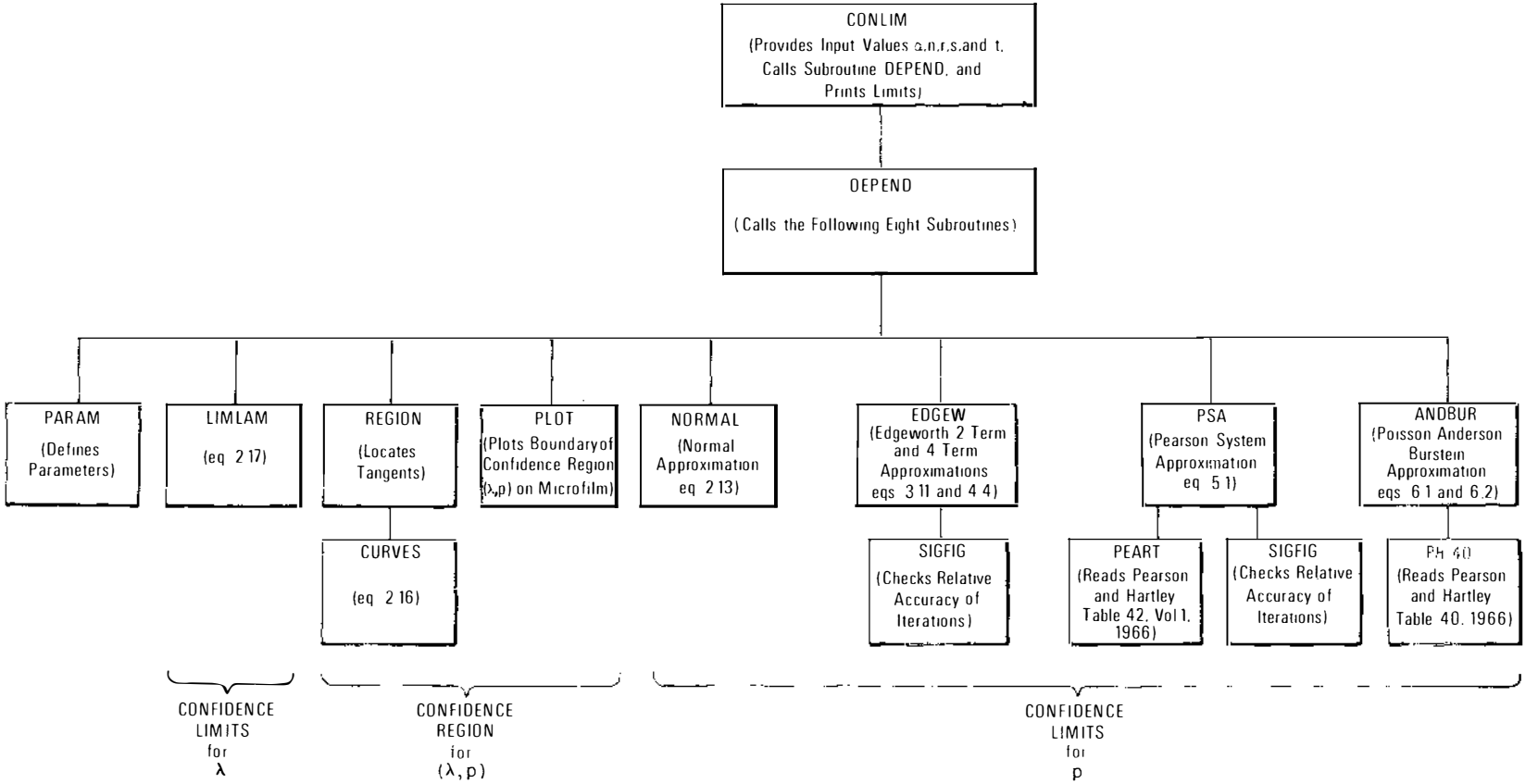

Figure 1. Flow diagram of computer program, CONLIM.

Subroutine REGION. This subroutine calculates the boundary and points of vertical and horizontal tangency for the normal approximation confidence region (2.16) of (λ,p). It divides the interval $0\leq\lambda\leq 1$ into ten equal subintervals and searches for the extent of the confidence region. This is done merely to utilize better the NPTS points that define the boundary (the tangents can be located more accurately than if the points were spread over the entire [0,1] interval). Of course, if more subdivisions are used the tangents can be determined more accurately.

The symbol KX denotes the number of points used for the microfilm plot of the boundary. Here KX=100.

Next, subroutine CURVES is called NPTS times to define the boundary and to locate the vertical and horizontal tangents. The two vertical tangents (limits on λ) are found by noticing when λ is first in and then out of the confidence region. The two horizontal tangents (limits on p) are found by noticing when, moving from $\lambda=0$ to $\lambda=1$, the slope of the upper boundary changes from positive to negative, and the slope of the lower boundary changes from negative to positive.

The coordinates (λ,p) of the lower boundary of the confidence region are denoted by (GP,GL) and those of the upper boundary by (GP,GU).

Subroutine CURVES. This subroutine defines the lower and upper boundary (PL and PU, respectively) of the confidence region for (λ,p). It will be called once for each of the NPTS values of λ. If $\hat{\lambda}=0$ (e.g. when r=0), we factor λ from equation 2.16. This is handled in the subroutine by going to statement 13 instead of statement 14.

Whenever λ lies beyond the confidence region, assign $PL=PU=-10^6$. This arbitrarily chosen, unrealizable value is used in subroutine REGION to locate the vertical tangents.

Subroutine PLOT. This subroutine plots the boundary of the confidence region for (λ,p). It utilizes the CDC-250 microfilm

recorder and subroutines peculiar to the NOAA CDC 6600 computer. Note that GL(KE), GU(KE), and GP(KE) are the KE values of the lower boundary, upper boundary, and abscissa. These are in the common block PLT in subroutine REGION. Users who must write their own plotting subroutine can access these values through

COMMON/PLT/GL(500), GU(500), GP(500), KE.

Subroutine NORMAL. This subroutine computes the normal approximation confidence limits for p, p_{L0}, and p_{U0} (eq. 2.13).

Subroutine LIMLAM. This subroutine computes the normal approximation confidence limits for λ (eq. 2.17).

Subroutine EDGEW. This subroutine determines the Edgeworth 2- and 4-term approximation confidence limits for p (Secs. 3-4). The lower and upper 2-term limits are PLI and PUI, respectively. The lower and upper 4-term limits are PPLI and PPUI, respectively (throughout this subroutine, the extra P in the 4-term equations is added to simulate the prime used in Section 4).

Initially these four limits are assigned the values obtained by the normal approximation. Each successive approximation, obtained by iteration, is checked with the previous one. Subroutine SIGFIG is called to determine if two successive values differ by less than a specified number (MINSIG) of significant figures (i.e., relative precision). A second test is made to see whether or not successive values differ by a specified number (MINDEC) of digits (i.e., absolute precision). If successive values of all four limits pass either test, iteration is halted. In any case, iteration is halted after MAXIT iteration. MINSIG, MINDEC, and MAXIT are specified in the main program.

Subroutine SIGFIG. This subroutine tests to determine if two consecutive iterative values (from either subroutine EDGEW or subroutine PSA) differ by less than JSIG significant figures.

The loop is satisfied when the values are multiplied by a power of 10 that is large enough to render the resulting value equal to an integer with JSIG digits.

Subroutine PSA. This subroutine determines the confidence limits for p from the Pearson system approximation (Section 5). The distribution of s can be represented by the Pearson system of distributions.

Subroutine PEART reads the tabular values, u_α, for the α=.025 or α=.050 percentage points of the Pearson distribution. One table is used for the lower limit and another for the upper limit. Locate $\beta_{1L,i-1}$ and $\beta_{2L,i-1}$ in the table. If a value is beyond the table, assign it the tabular value nearest it. Since the table is two dimensional, we use double interpolation to determine $u_{\alpha_{Li}}$ (similarly for $u_{\alpha_{Ui}}$). Then $u_{\alpha_{Li}}$ is substituted into p_{Li} (eq. 3.11).

Iteration proceeds precisely as it does in subroutine EDGEW (see above).

Subroutine PEART. This subroutine uses the data cards which contain (2 decimal places) the upper and lower Pearson and Hartley Table 32 (vol. II, 1972) for α=.025 and α=.050. It manipulates the data from the cards to resemble the tables (for interpolation). It then extends the tables, both up and down, by assigning as unlisted values the nearest value listed, either above or below. [Pearson and Hartley's Table 32 (1972) has been used to extend their Table 42 (1966), which is not indicated in Figure 1.]

Subroutine ANDBUR. This subroutine computes the modified Poisson-Anderson-Burstein approximate confidence limits for p (Section 6). The lower and upper confidence limits for independent trials, SLOWER and SUPPER, are provided by subroutine PH40. These values are then substituted in (6.1) and (6.2) to determine the modified Poisson-Anderson-Burstein confidence limits, p_L and p_U.

Subroutine PH40. This subroutine contains the 80, 90, 95, and 99% confidence limits for the mean of a Poisson distribution (independent errors) (Pearson and Hartley, 1966, Table 40; Crow, 1974, Table 1). It then interpolates if there are between 30 and 100 errors. The table cannot be used if s>100. The lower and upper limits, called SLOWER and SUPPER, are used in subroutine ANDBUR.

9. APPLICATION TO COX-LEWIS TELEPHONE DATA

The calculation of all of the confidence intervals for error probability, p, of the confidence interval for the conditional error probability, λ, and of the confidence region for (λ,p) derived in Sections 2-6 is illustrated here on telephone data. The calculations are easily, though a little tediously for the Edgeworth and Pearson, performed with a pocket calculator, and the steps are recorded here for checking by the interested reader. However, the steps are unnecessary for those in possession of the computer program described in Section 8.

Cox and Lewis (1966, pp. 256-257) tabulate the intervals between successive errors in a sequence of telephone messages, which was "obtained in joint work by IBM Germany and the German Postal Administration." Slightly over one million characters were transmitted and received, but the first 20,000 will be used for this example. It will be assumed that the error probability is constant for all characters. Since only the intervals are listed, it is not known whether the first transmission is in error or not, but it is assumed to be correct. There are then 38 errors and 13 pairs of successive errors in the 20,000 transmissions. In the notation of Section 2,

$$n = 20{,}000,\ r = 13,\ s = 38,\ \text{and}\ t = 0.$$

From (2.1),

$$\hat{p} = 38/20{,}000 = .0019,\ \hat{q} = 1-\hat{p} = .9981.$$

From (2.2) and (2.8),

$$\hat{\lambda} = .342097,\ \hat{\rho} = .340844.$$

We shall calculate 90% confidence intervals, so that α=.05 and u_α=1.64485 [from Abramowitz and Stegun (1964), Table 26.5, for example].

The computer program described in Section 8 gives immediately the following upper and lower confidence limits for p:

	Anderson-Burstein	Normal	Edgeworth 2	Edgeworth 4	Pearson
p_U	.0027406	.0028023	.0027580	.0027633	.0027537
p_L	.0012207	.0012817	.0012451	.0012517	.0012517

These agree remarkably well, to 4 or 5 DP and 2 or 3 SF, due to the large sample size and substantial values of s and r. The program also gives the normal approximation limits (90%) on λ by itself (2.17),

$$.22984 < \lambda < .47535,$$

and the coordinates of the vertical and horizontal tangents of the 90% (joint) confidence region for (λ,p):

(λ=.4500, p=.00365)

(.1802, .00148) (.5237, .00234)

(.3000, .00114)

The confidence region will be discussed further after the pocket calculator results are obtained step by step.

The modified Anderson-Burstein limits (6.1) or (6.2) are the easiest to obtain. By linear interpolation in Table 1 of OT Report 74-51 (Crow, 1974), U=49.6, L=28.48. By the formulas given there the confidence limits under the assumption of independent transmissions are

$$p_{UI} = \frac{U}{n+(U-s)/2} = \frac{49.6}{20,005.8} = .002479,$$

$$p_{LI} = \frac{L}{n-(s-1-L)/2} = \frac{28.48}{19,995.7} = .001424.$$

(These are guaranteed to be accurate to only 2 SF, but more figures are carried to avoid roundoff error in the final result.) Hence by (6.1)

$$p_U = .001900 + .000579 \times 1.4262 = .002726 = .0027,$$

$$p_L = .001900 - .000476 \times 1.4262 = .001221 = .0012,$$

and it is not necessary to use (6.2). Slight differences between the hand calculation using Table 1 of OTR 74-51 and the computer program results are possible because the program incorporates the more accurate Poisson confidence limits of Pearson and Hartley (1966), while Table 1 is purposely rounded because the method is designed for simplicity to guarantee only 2-digit accuracy.

Proceeding to the normal approximation, we first calculate, from (2.8), $\hat{V}$=40,604.8. Then we have immediately the normal approximation limits from (2.13):

$$p_{U0} = (8\times10^8)^{-1}[40{,}604.8\times1.64485^2+77\times20{,}000 + \{(40{,}604.8\times1.64485^2+77\times20{,}000)^2-(77\times20{,}000)^2\}^{\frac{1}{2}}]$$

$$= .0028023,$$

$$p_{L0} = .0012817.$$

The normal approximation 90% confidence limits for λ are, from (2.17),

$$\lambda_{U0} = .47535, \quad \lambda_{L0} = .22984.$$

Somewhat more quickly obtained but less accurate normal limits for λ are available from (2.18), which gives .46869 and .21551 in this case, respectively 1.4% and 6.2% less than the (2.17) results. The use of (2.18) is not recommended except for a quick, temporary calculation. However, considering the width of the confidence interval, one would probably be satisfied with 2-digit results anyway. Even the (2.17) results are probably accurate to no more than 3 digits.

The Edgeworth two-term approximation (3.11) is initiated by calculating (3.4) from the normal approximation limits p_{U0} and p_{L0}:

$$p_{U0} = .340248, \qquad \rho_{L0} = .341252,$$
$$V_{U0} = 40{,}513.4, \qquad V_{L0} = 40{,}667.5,$$
$$B_{U0} = .00230626, \qquad B_{L0} = .00231509.$$

From (3.10) and (3.7)

$$\alpha_{U1} = .0576634, \quad \alpha_{L1} = .0386251.$$

Hence from (3.11)

$$p_{U1} = .0027571, \quad p_{L1} = .0012461.$$

These differ from p_{U0} and p_{L0} in the third significant figure (SF), so we iterate (3.4)-(3.11):

$$\rho_{U1} = .340278, \qquad \rho_{L1} = .341276,$$
$$V_{U1} = 40{,}518.0, \qquad V_{L1} = 40{,}671.1,$$
$$B_{U1} = .00230643, \qquad B_{L1} = .00231529,$$

$$\alpha_{U2} = .0575093, \qquad \alpha_{L2} = .0383288,$$
$$p_{U2} = .0027580, \qquad p_{L2} = .0012451.$$

Examining the sequences p_{U0}, p_{U1}, p_{U2} and p_{L0}, p_{L1}, p_{L2} indicates that the last iterates are accurate (as far as the convergence to the Edgeworth two-term limits is concerned) to at least 4SF and 6 decimal places (DP).

The Edgeworth four-term approximation could be initiated using the values calculated from p_{U0} and p_{L0} also, but we have presumably better values from the two-term calculations above. Hence we start with

$$p'_{U0} = .002758, \qquad p'_{L0} = .001245,$$
$$\rho'_{U0} = .340277, \qquad \rho'_{L0} = .341277,$$
$$V'_{U0} = V_{U1}, \qquad V'_{L0} = V_{L1},$$
$$B'_{U0} = B_{U1}, \qquad B'_{L0} = B_{L1}.$$

From (4.3) $C'_{U0} = .0000104995$, $C'_{L0} = .0000106129$. Substituting in (4.4) and using the Abramowitz and Stegun (1964) Table 26.1 gives

$$\alpha'_{U1} = .0565096, \quad \alpha'_{L1} = .0402368.$$

Then from (3.11)

$$p'_{U1} = .0027635, \quad p'_{L1} = .0012515.$$

These agree with p'_{U0} and p'_{L0} to 3SF, but we confirm with another iteration:

$$\rho'_{U1} = .340274, \qquad \rho'_{L1} = .341273,$$
$$V'_{U1} = 40{,}517.4, \qquad V'_{L1} = 40{,}670.6,$$
$$B'_{U1} = .00230648, \qquad B'_{L1} = .00231528,$$
$$C'_{U1} = .0000104992, \qquad C'_{L1} = .0000106128,$$
$$\alpha'_{U2} = .0565425, \qquad \alpha'_{L2} = .0402871,$$
$$p'_{U2} = .0027633, \qquad p'_{L2} = .0012516.$$

The Pearson system approximation could be initiated using the values calculated from p_{U0} and p_{L0} also, but the Edgeworth four-term limits are presumably better and provide a running start in (5.1):

$$B_{U0} = B'_{U1}, \qquad B_{L0} = B'_{L1},$$
$$C_{U0} = C'_{U1}, \qquad C_{L0} = C'_{L1},$$
$$\beta_{1U0} = .06931, \qquad \beta_{1L0} = .15419,$$
$$\beta_{2U0} = 3.09119, \qquad \beta_{2L0} = 3.20351.$$

From Table 42 of Pearson and Hartley (1966, Vol. I) [or Table 32 of Pearson and Hartley (1972, Vol. II)

$$u_{\alpha U1} = 1.568, \quad u_{\alpha L1} = 1.752.$$

Then from (3.11)

$$p_{U1} = .0027530, \qquad p_{L1} = .0012503.$$

Iteration does not change these values, to the limited accuracy available from Table 42. Table 32 is more accurate, as well as having a larger range of values of β_1 and β_2, but the accuracy of Table 42 is adequate. Thus the Pearson system confidence limits for p, either those immediately above or those shown earlier from the computer program, are probably accurate just to 3SF (5DP) simply due to using a table of Pearson system percentage points with 2DP. [Although Table 32 has 4DP, it was replaced in the computer program by a table with 2DP and uniform increments in β_1 (by interpolation) to conform with the Table 42 previously incorporated in the program.]

This completes the calculation of 90% confidence limits for p by itself and confirms the limits from the computer program tabulated early in this section. Noting the essentially constant length of the intervals, we conclude that any one pair gives a satisfactory 90% interval, and we state that $.00125<p<.00276$.

The boundary of the 90% confidence region for (λ,p) is obtained from (2.16) by substituting $\hat{p}=.0019$, $\hat{q}=.9981$, $\hat{\lambda}=.342097$, and the upper 10% point of chi-squared with 2 degrees of freedom, $\chi^2_{2,.10}=4.60517$ (Harter, 1964, but any standard table will do):

$$p^2[\lambda(1-\lambda)^2+1.9962\lambda(1-\lambda)(.342097-\lambda)+.9981(.9962+\lambda)(.342097-\lambda)^2]$$
$$-p\lambda(1-\lambda)[.0038(1-\lambda)+.00379278(.342097-\lambda)+.000229821(.9962+\lambda)]$$
$$+.0000361\lambda(1-\lambda)^2 = 0.$$

Substituting λ=.3 yields

$$1669.398526p^2 - 6.546873p + .0053067 = 0,$$

so two points on the boundary are

$$(\lambda,p) = (.3, .0011447), (.3, .0027770).$$

Other points are found similarly:

λ	p_1	p_2	Discriminant
.35	.0011687	.0031639	
.40	.0012294	.0034964	
.45	.0013515	.0036500	
.50	.0016489	.0032801	1.85233
.55	complex	complex	-2.13405
.52	.0020197	.0027085	0.29655
.53	complex	complex	-0.50561
.5235	.0022567	.0024239	0.17202

Therefore by interpolation the extreme λ is .5237 to 4SF, occurring at p=.00234, compared with the upper 90% limit λ_{U0}=.47535 on λ by itself. The left-hand part of the confidence region is likewise calculated:

λ	p_1	p_2	Discriminant
.25	.0011570	.0023636	
.20	.0012491	.0018731	1.51879
.15	complex	complex	-2.24046
.18	complex	complex	-0.014255
.1803	.0014569	.0015038	0.008748

Therefore the lower extreme λ is .1802 to 4SF at p=.00148, compared with the lower 90% limit λ_{L0}=.2298 on λ by itself. A few more points are desirable to ascertain the extreme values of p:

λ	p_1	p_2
.48	.0014891	.0035315
.46	.0013892	.0036357
.44	.0013166	.0036470

The resulting 90% confidence region is plotted in figure 2, along with the 90% confidence limits for λ by itself and the various 90% limits for p by itself.

We see from Figure 2 that the smallest value of p of points in the confidence region is .00114, occurring at λ=.30, and that the largest such value is .00365, occurring at λ=.44. Since we are 90% confident that the region contains (λ,p), we are *a fortiori* at least 90% confident that p is between .00114 and .00365, which are thus conservative 90% confidence limits on p. We already have what appear to be excellent approximate limits for p, but they were calculated assuming that $\hat{\lambda}$=.342097 is a satisfactory approximation for the unknown λ. The assumption is reasonable because $\hat{\lambda}$ is a consistent and asymptotically normal estimate of λ and will tend to be too small about as often as it is too large, so that the confidence interval will be "too short" about as often as it is "too long" in different samples. This intuitive argument is not fully confirmed by the Monte Carlo simulation in Section 11, $\hat{\lambda}$ being found to have negative bias. The confidence region for (λ,p) provides reassurance about the confidence interval for p alone because it assumes neither $\lambda=\hat{\lambda}$ nor a prior value of λ.

Thus we can view the 90% confidence limits on p alone as providing a 90% confidence *region* for (λ,p) also, as indicated by the dashed lines in Figure 2, the boundary of the region in the λ direction being the line segments at λ=0 and λ=1. The width of this region in the p direction is smaller than that of the oval by virtue of relaxing the claim on λ. It is easy to imagine that both regions have a "confidence content" of 90%.

Alternatively the same argument could be made for the 90% confidence limits on λ. Since $\hat{p}$ and $\hat{\lambda}$ are correlated, the rectangular intersection of the two 90% rectangles is not an 81% confidence region for (λ,p). Likewise, our oval 90% region for (λ,p) is not rigorously comparable with the rectangle obtained by calculating $100(.90)^{\frac{1}{2}}$=94.9% confidence intervals for p and for λ separately. However, the rectangle obtained as the

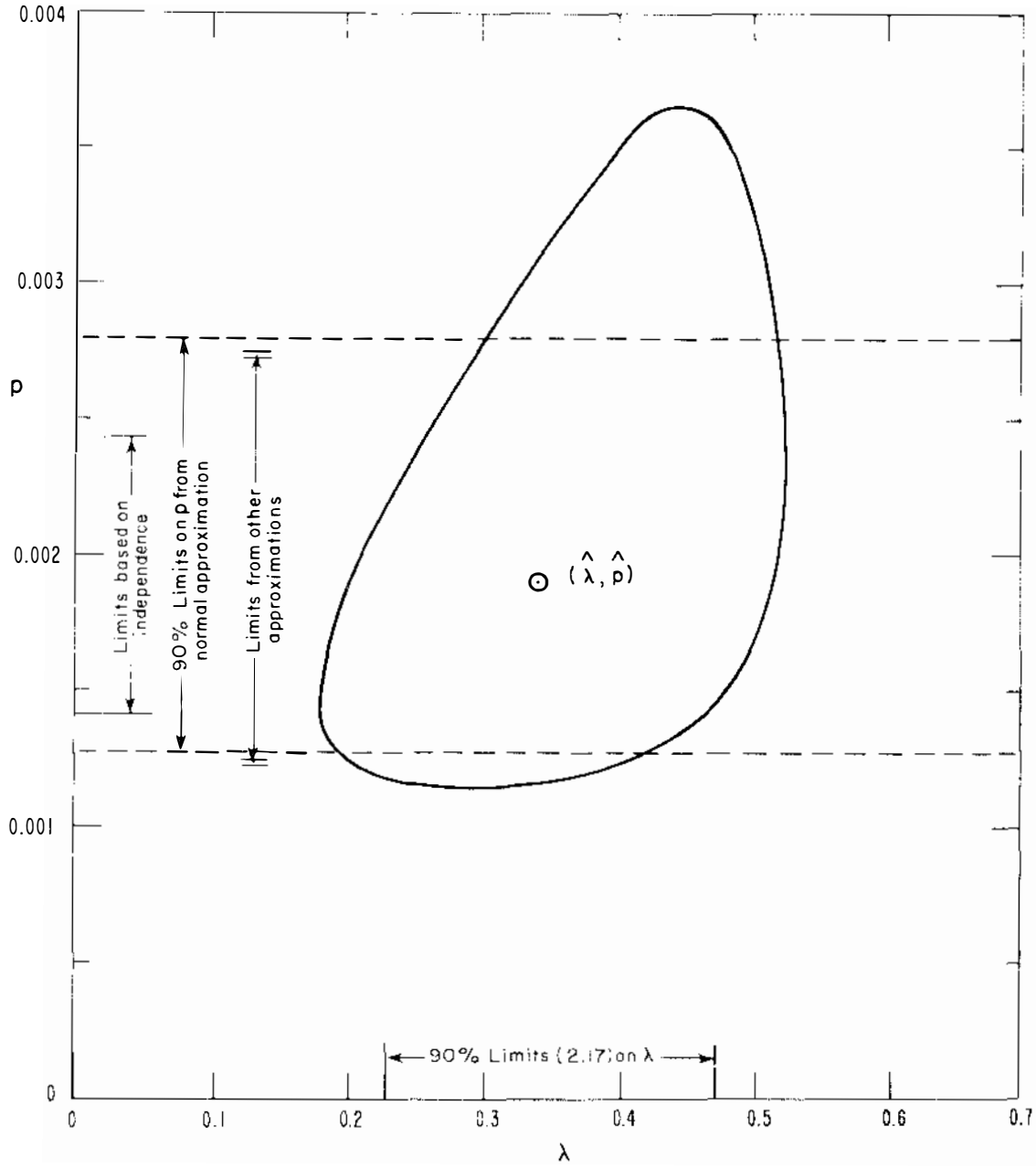

Figure 2. 90% confidence region for (λ,p) from first 20,000 Cox-Lewis telephone data and normal approximation (38 errors, $\hat{p}$=.0019, $\hat{\lambda}$=.3421).

intersection of separate 95.0% confidence intervals for p and for λ is a rigorous confidence region for (λ,p) with confidence level at least 90.0%. More generally, the rectangle obtained as the intersection of a $1-\alpha_1$ confidence interval for λ and a $1-\alpha_2$ confidence interval for p is a confidence region for (λ,p) with confidence level at least $1-\alpha_1-\alpha_2$. This follows from the simplest "Bonferroni Inequality" (Feller, 1968, p. 110; Dunn, 1974).

The Cox-Lewis data are also used as the example for the computer program in Appendix B, where the limits are slightly wider because $\tilde{\lambda}$ is used rather than $\hat{\lambda}$.

10. TESTING THE VALIDITY OF INDEPENDENCE AND MARKOV CHAIN MODELS

10.1. Introduction

The present development of confidence limits for the error rate p was motivated by the obvious departure of the pattern of errors in transmission from the classical Bernoulli model of independent trials with constant error rate. What justification is there that the model adopted herein of a stationary first-order Markov chain is a satisfactory model? This model has just two parameters, p and λ, whereas several studies (Gilbert, 1960; Elliott, 1963; Fritchman, 1967) have indicated the need for as many as five or six parameters to fit digital communications data fully. Nevertheless, there is considerable justification for the adopted model:

(1) The added parameter λ provides a measure of dependence completely lacking in the Bernoulli model, so the confidence intervals should be considerably better approximations then those from the Bernoulli model, if there is dependence. Since the Bernoulli model is a special case with $\lambda=p$, there is no appreciable harm done if trials are actually independent except complication in computing the limits.

(2) There is no "true" model of any physical phenomenon, only closer and closer approximations. The (λ,p)

model may be likened to fitting an inclined straight line to data for which a horizontal straight line would be a first approximation analogous to the model of independent trials.

(3) In the course of calculating the (Poisson-)Anderson-Burstein-type approximation for confidence limits, one gets the limits based on independence also and can thus observe how different the two pairs of limits are. It is reasonable to conclude that going to a more complex model, such as a second-order Markov chain, would introduce less change than going from zeroth-order (independent trials) to first-order chain, and often the change would be negligible in practice.

(4) Except for the Anderson-Burstein approximation, the formulas for confidence limits presented herein are already fairly messy, and a higher-order approximation could be expected to be even messier and hence perhaps beyond practical interest.

(5) The finite sample size of data presents only a limited amount of information, which may be insufficient to calculate the further parameters of a more complex model, or at least to calculate them with useful precision. It is already impossible to calculate $\hat{\lambda}$ in (2.2) or λ^* in (2.4) if no errors are observed. The effective sample size for estimating λ is the number of errors, s, not the number of transmissions, n [cf. equation (12.5)]. Similarly, the effective sample size for estimating a second-order conditional probability would be the number of pairs of errors, r. Thus the amount of information available for further parameters tends to decrease geometrically.

If it is believed prior to the experiment that the errors may occur independently and the data obtained are consistent with that hypothesis, then the classical binomial confidence

limits of OT Report 74-51 (Crow, 1974), should be used. An easy test of independence, the total number of runs, is described in Section 10.2. If the hypothesis is rejected by that test, then the confidence limits of the present report will probably be satisfactory, but the assumption of a stationary first-order Markov chain should be tested by one or more of the three tests outlined in Section 10.3 (Pearson X^2, likelihood ratio statistic G^2, sum of squared Freeman-Tukey deviates FT^2).

Going to a first-order or higher-order Markov chain is hardly the only possible generalization of independent trials. Gilbert (1960), Elliott (1963), Fritchman (1967), and others have introduced further models. A communication system might depart from independent trials with constant error rate by changes of state with different error rates in different states. Gilbert and Elliott considered such models in which the changes occurred at random so that the model still represents a stationary process. The Klotz model of this report is the special case of the Gilbert model in which the transition probability from a good state, G, to a bad state, B, is the same as that from B to G.

It is possible that changes of regime might occur at systematic rather than random points in time, such as sunset and sunrise or changes of personnel shifts. Within each regime the stationary first-order Markov model of this report may apply, and large-sample tests for differences between the values of $\hat{p}$ and $\hat{\lambda}$ can be made in a standard way based on the asymptotic normality of distributions, as discussed in Section 10.4.

10.2. Run Test of Independence

Consider the following sequence of bits

0,0,0,1,0,1,1,1,0,0,0,0,1,1

in which the occurrence of an error is denoted by a 1. A sequence of k identical symbols that is preceded and followed by a different symbol or no symbol is called a run of length k

(Dixon and Massey, 1951, p. 254). Thus, in the above sequence there are 2 runs of length 1, 1 of length 2, 2 of length 3, and 1 of length 4. If there is dependence between transmissions such that errors tend to occur together, then there will tend to be fewer runs than with independent transmissions. (In other situations there might be more runs than with independence.) Consequently a test of independence can be based on the total number of runs. In the above sequence the total number of runs is 6.

The total number of runs, say u, in a sample of independent transmissions is influenced by the number of 0's, say N_0, and the number of 1's, say N_1. For small values (20 or less) of N_0 and N_1 the critical numbers of runs beyond which the hypothesis of independence is rejected are tabulated (Dixon and Massey, 1951, Table 11), but for the large samples needed in communications systems the normal approximation of the distribution of u is needed and is satisfactory. The mean and variance of u are (Dixon and Massey, 1951, p. 256)

$$\mu_u = \frac{2N_0N_1}{N_0+N_1} + 1,$$

$$\sigma_u^2 = \frac{2N_0N_1(2N_0N_1-N_0-N_1)}{(N_0+N_1)^2(N_0+N_1-1)} .$$

Thus, if the observed number of runs were less than $\mu_u+2.326\sigma_u$, we would reject the hypothesis of independence at the 1% significance level (one-sided test).

10.3. Test of the Order of a Markov Chain

Testing whether a sequence of correct and incorrect transmissions can be represented by a Markov chain of first order, or any order for that matter, is a special case of the problem of testing the goodness of fit of a general Markov chain, which has been discussed extensively in the periodical statistical

literature (e.g., Bartlett, 1951; Hoel, 1954; Anderson and Goodman, 1957; Billingsley, 1961; Guthrie and Youssef, 1970; Yakowitz, 1976). However, perhaps the best exposition for application is in the book by Bishop, Fienberg, and Holland (1975), in Sections 7.2-7.4, especially Example 7.4-1.

The method is similar to that of testing independence in a 2x2 contingency table: The observed frequencies are compared with the expected frequencies under the null hypothesis of independence and a measure of the combined differences, Pearson X^2 or the likelihood ratio in particular, is calculated and compared with a tabulated chi-squared percentage point. For testing whether a (Markov) chain is of first order it is assumed that the data come from no more than a second-order chain. Notation and formulas will be given first for that case but generalize to any order. They will be illustrated on the telephone data tabulated by Cox and Lewis (1966) and used in Section 9.

We consider only two states, 0 or 1, and a single sequence of n transmissions, for most of which we expect a 0 to be recorded, indicating a correct bit is received, while 1 indicates an error. There will be n-1 one-step transitions of the four types 00, 01, 10, 11, and n-2 two-step transitions of the eight types 000,001,...,111. Let x_{ijk} denote the number of transitions of type ijk, where i, j, and k each take on the value 0 or 1. Let the sum of x_{ijk} over any subscript be denoted by replacing that subscript by +. For example,

$$x_{11+} = x_{110} + x_{111}.$$

Similarly the sum over two subscripts is denoted with two +;s, e.g.,

$$x_{1++} = x_{10+} + x_{11+} = x_{100} + x_{101} + x_{110} + x_{111} = \sum_{j,k} x_{ijk}.$$

Finally, the sum over all subscripts is denoted by x_{+++} and must equal n-2.

The likelihood ratio statistic is

$$G^2 = 2 \sum_{i,j,k} x_{ijk} \ln \frac{x_{ijk}}{\hat{m}_{ijk}} , \tag{10.1}$$

where $\hat{m}_{ijk}$ is the expected frequency calculated from the marginal sums and proportions with the same one-step transitions, that is, under the null hypothesis that the process is first order,

$$\hat{m}_{ijk} = x_{ij+} \frac{x_{+jk}}{x_{+j+}} . \tag{10.2}$$

When the null hypothesis is true and the $\hat{m}_{ijk}$ are "not too small," say not less than 2, G^2 is distributed approximately as chi-squared with 2 degrees of freedom (d.f.) (in this case of the two-state chain of second order or less) and may be judged therefore by comparison with the tabulated χ^2 percentage point. If the null hypothesis is not true, x_{ijk} and $\hat{m}_{ijk}$ will tend to differ more and G^2 will tend to be larger than the tabulated percentage point. However, as illustrated in the example below, the chi-squared distribution may be a poor approximation in the case of some telecommunications data.

The Pearson X^2 statistic is

$$X^2 = \sum_{i,j,k} \frac{(x_{ijk} - \hat{m}_{ijk})^2}{\hat{m}_{ijk}} . \tag{10.3}$$

The statistic X^2 is also asymptotically distributed as chi-squared with 2 d.f. when the chain is in fact of first order.

Finally, the sum of squared Freeman-Tukey deviates (Bishop et al., 1975, p. 137) is

$$FT^2 = \sum_{i,j,k} (\sqrt{x_{ijk}} + \sqrt{x_{ijk}+1} - \sqrt{4m_{ijk}+1})^2 . \tag{10.4}$$

It also is asymptotically distributed as chi-squared with 2 d.f. when the chain is in fact of first order. The approach to the limiting distribution may be quite different for the three statistics defined above.

Example. Cox and Lewis (1966) tabulate the intervals between successive errors in a sequence of telephone messages,

which was "obtained in joint work by IBM Germany and the German Postal Administration." Slightly over one million characters were transmitted, but the first 20,000 will be used for this example. Since only the intervals are listed, it is not known whether the first transmission is in error or not. If we take it as correct, there are 38 errors, 13 pairs of successive errors, and 5 triples of successive errors. Counting the other triples also, we found the data in Table 3. Thus the numbers of one-step transitions are the same whether arrived at as x_{ij+} or x_{+jk}, but both are shown for exposition and direct substitution in (10.2). The expected cell frequencies (10.2) under the first-order hypothesis are then

$$\hat{m}_{111} = 13x13/38 = 4.447, \quad \hat{m}_{101} = 25x25/19,960 = .03131,$$
$$\hat{m}_{011} = 25x13/38 = 8.553, \quad \hat{m}_{001} = 19,935x25/19,960 = 24.969,$$
$$\hat{m}_{110} = 8.553, \quad \hat{m}_{100} = 24.969,$$
$$\hat{m}_{010} = 16.447, \quad \hat{m}_{000} = 19,910.031.$$

We note that there are only two independently determined expected frequencies among these eight, consistent with the statement that the limiting chi-squared distribution has 2 d.f. under the null hypothesis. One of these expected frequencies, $\hat{m}_{101}$, is drastically less than the minimum frequency of 2 for the approximating chi-squared distribution to hold, but we shall calculate G^2, X^2, and FT^2 anyway for lack of something better. It would thus be desirable to have a limit approximation for which a single expected cell frequency is allowed to be arbitrarily small (cf. Bishop et al., 1975, p. 140; Yarnold, 1970).

From (10.1)

$$G^2 = 2[5 \ln 5/4.447 + 2 \ln 2/.03131 + 8 \ln 8/8.553 + 23 \ln 23/24.969 + 8 \ln 8/8.553 + 23 \ln 23/24.969 + 17 \ln 17/16.447 + 19,912 \ln 19,912/19,910.031]$$
$$= 13.167.$$

Table 3. Numbers of Two-step and One-step Transitions in the First 20,000 Telephone Transmissions Tabulated by Cox and Lewis (1966)

Two-step Transitions:

	Position t			
	1		0	
	Position t-1		Position t-1	
Position t-2	1	0	1	0
1	$x_{111}=5$	$x_{101}=2$	$x_{110}=8$	$x_{100}=23$
0	$x_{011}=8$	$x_{001}=23$	$x_{010}=17$	$x_{000}=19,912$

One-step Transitions:

	Position t	
Position t-1	1	0
1	$x_{+11}=13$	$x_{++0}=25$
0	$x_{+01}=25$	$x_{+00}=19,935$

	Position t-1		
Position t-2	1	0	
1	$x_{11+}=13$	$x_{10+}=25$	$x_{1++}=38$
0	$x_{01+}=25$	$x_{00+}=19,935$	$x_{0++}=19,960$
	$x_{+1+}=38$	$x_{+0+}=19,960$	$x_{+++}=19,998$

The tabulated chi-squared percentage points for 2 d.f. at the 0.5% and 0.1% levels are $\chi^2_{2,.005}$=10.597 and $\chi^2_{2,.001}$=13.816. Hence, according to the G^2 statistic, the data are not consistent (at the 0.5% significance level) with the hypothesis of a first-order process.

Similarly, from (10.3),

$$X^2 = 124.2 \text{ with 2 d.f.}$$

Comparing this with the same χ^2_2 percentage points as above, we see that the X^2 statistic rejects the first-order hypothesis even more strongly than G^2.

From (10.4),

$$FT^2 = 4.776 \text{ with 2 d.f.},$$

whereas $\chi^2_{2,.1}$=4.605, $\chi^2_{2,.05}$=5.991. Hence, according to the FT^2 statistic, the hypothesis that the process is of first order would not be rejected at the often used 5% significance level, though it would be, barely, at the 10% level.

Thus the three different criteria for judging the first-order model give widely differing results, but FT^2 may be the most reliable from the evidence presented by Freeman and Tukey (1950) and the fact that it was specifically designed for small (Poisson) frequencies.

It is also instructive to compare the individual observed and expected frequencies and the individual terms in G^2, X^2, and FT^2. The term in m_{101}=0.03131 accounts for 124.1 of χ^2, 4.349 of FT^2, and 16.628 of G^2! (G^2 has three negative terms also, two of them -3.778). Thus <u>all</u> of the apparent departure from the first-order model is accounted for by an observed frequency of 2 where 0.03131 is "expected" (on the average). The Poisson distribution probably applies quite well to this particular cell; for it the probability of a 0 count is exp(-0.03131)=0.96917; of 1, 0.03034; and 2 or more, 0.00048. This confirms that the observed frequency of 2 is inconsistent with the first-order model.

There are two conclusions to all of this calculation and discussion:

(1) The first 20,000 transmissions of the Cox-Lewis data are inconsistent with a first-order Markov chain model.

(2) The formal criteria, G^2, X^2, and FT^2, given for judging goodness of fit in large samples cannot be applied unquestioningly to telecommunications or other data where probabilities or error rates may be very small.

The logical first hypothesis to test would have been that of independence (i.e., zeroth order), but we discussed the test of first order first because it is of most interest, there being no doubt, usually and in this case in particular, of the lack of independence. Independence can be tested by the standard X^2 test of a 2x2 table, but that can also be seen to be the same as (10.2)-(10.3) with one subscript dropped. There is also only 1 d.f. with the marginal totals fixed. Statistics (10.1) and (10.4) can also be applied. In the 2x2 table at the bottom of Table 1, the expected frequencies under independence are

$$\hat{m}_{11+} = x_{1++}x_{+1+}/x_{+++} = 0.07221, \quad \hat{m}_{10+} = 37.92779,$$

$$\hat{m}_{01+} = 37.92779, \quad \hat{m}_{11+} = 19{,}922.07221.$$

Hence

$$G^2 = 119.2, \quad X^2 = 2147.1, \quad FT^2 = 48.8.$$

These are all much larger than the corresponding values for testing the first-order model (and are furthermore to be compared with the smaller χ^2 percentage point with just 1 d.f.). One would expect this; if the process is not of first order, it is a fortiori not of zeroth order, i.e., not a sequence of independent random variables.

We proceed naturally to testing whether the process is a second-order Markov chain, given that it is of not more than third order. It is necessary to go to the original Cox-Lewis data and tabulate the numbers x_{ijkl} of all of the different three-step transitions, as in Table 4. The total number of three-step transitions is $x_{++++} = 19{,}997$, which is 1 less than the number of two-step transmissions in Table 3. The last

Table 4. Numbers of Three-step Transitions in the First 20,000 Transmissions Tabulated by Cox and Lewis (1966)

Observed:

$x_{1111} = 2$	$x_{1011} = 2$	$x_{1101} = 0$	$x_{1001} = 0$
$x_{1110} = 3$	$x_{1010} = 0$	$x_{1100} = 8$	$x_{1000} = 23$
$x_{0111} = 3$	$x_{0011} = 6$	$x_{0101} = 2$	$x_{0001} = 23$
$x_{0110} = 5$	$x_{0010} = 17$	$x_{0100} = 15$	$x_{0000} = 19,888$

Expected:

$\hat{m}_{1111} = 1.923$	$\hat{m}_{1011} = 0.640$	$\hat{m}_{1101} = 0.640$	$\hat{m}_{1001} = 0.02654$
$\hat{m}_{1110} = 3.077$	$\hat{m}_{1010} = 1.360$	$\hat{m}_{1100} = 7.360$	$\hat{m}_{1000} = 22.973$
$\hat{m}_{0111} = 3.077$	$\hat{m}_{0011} = 7.360$	$\hat{m}_{0101} = 1.360$	$\hat{m}_{0001} = 22.973$
$\hat{m}_{0110} = 4.923$	$\hat{m}_{0010} = 15.640$	$\hat{m}_{0100} = 15.640$	$\hat{m}_{0000} = 19,888.027$

four transmissions were in fact correct, so the only change in two-step and one-step transitions in Table 3 is a reduction by 1 of those numbers involving only 0's: x_{000+}=19,911, x_{+00+}=19,934, x_{+0++}=19,959. The expected frequencies are calculated by the analogue of (10.2):

$$\hat{m}_{ijk\ell} = x_{ijk+} \frac{x_{+jk\ell}}{x_{+jk+}} . \qquad (10.5)$$

The statistics G^2, X^2, and FT^2 are calculated from the analogues of (10.1), (10.3), and (10.4) with summation over four subscripts, so that there are 16 terms. In G^2, terms involving $x_{ijk\ell}$=0 are taken to have the limiting value 0. Calculations from Table 2 give

$$G^2 = 6.628, \quad X^2 = 5.677, \quad FT^2 = 5.628.$$

We note in Table 4 that not less than 3 of the 16 expected frequencies are independently determined. Actually 4 are independently determined, the coincidence being an artifact of the particular observed frequencies. The theory shows that, in testing whether a chain given to be third order may be second order, the d.f. of the limiting chi-squared distribution is 4. Thus the above values of G^2, X^2, and FT^2 are to be compared with percentage points of χ^2_4; $\chi^2_{4,.1} = 7.779$, so the data are consistent with the hypothesis of second order even at the 10% significance level, according to all three statistics.

It is noteworthy that G^2, X^2, and FT^2 are about equal under the second-order hypothesis, whereas they differ greatly under the first-order and zeroth-order hypothesis. This illustrates the general theory that all three statistics are asymptotically distributed as chi-squared with the same d.f. when the null hypothesis is true, but not otherwise. The approximate equality provides some reassurance about comparing them with chi-squared when some expected frequencies are so far below the usually designated minimum allowable.

In the same way, a Markov chain with two states (0 and 1) of any order r can be tested for consistency with the hypothesis

that it is of order r-1. The only essential change is the number of degrees of freedom of the limiting chi-squared distribution, which is 2^{r-1} (Hoel, 1954). Note that a chain of order r is described with r+1 subscripts.

Although the Cox-Lewis data are inconsistent with a first-order model, the points cited in Section 10.1 can be invoked to justify using the confidence limit approximations of this report based on the first-order model.

10.4. Test of Nonstationarity

If it is believed that the error rate p and the conditional error rate λ may change during the acquisition of data, the data should be separated into subsamples considered to be homogeneous and tested for differences between the subsamples. Only a large-sample test based on the asymptotic normality of $(\hat{\lambda},\hat{p})$ and their asymptotic variance-covariance matrix (2.14) will be given (and indeed an exact test can hardly be expected). In the use of (2.14), it is assumed that within each subsample the first-order Markov chain model applies.

Suppose that in a single long sample of size n there are k homogeneous subsamples of size $n_1,n_2,\ldots,n_k$ and estimates $(\hat{p}_1,\hat{\lambda}_1)$, $(\hat{p}_2,\hat{\lambda}_2),\ldots,$ $(\hat{p}_k,\hat{\lambda}_k)$. Under the null hypothesis that all the p_i are equal and all the λ_i are equal, we may estimate the common values of p and λ by the $\hat{p}$ of (2.1) and the $\hat{\lambda}$ of (2.2). Consequently, from (2.14), for large samples, the variance-covariance matrix of each $(\hat{\lambda}_i,\hat{p}_i)$ can be estimated by

$$\begin{pmatrix} s^2_{\hat{\lambda}_i} & s_{\hat{\lambda}_i\hat{p}_i} \\ s_{\hat{\lambda}_i\hat{p}_i} & s^2_{\hat{p}_i} \end{pmatrix} = \begin{pmatrix} \hat{\lambda}(1-\hat{\lambda})/(n_i\hat{p}) & \hat{q}\hat{\lambda}/n_i \\ \hat{q}\hat{\lambda}/n_i & \hat{p}\hat{q}(1-2\hat{p}+\hat{\lambda})/[n_i(1-\hat{\lambda})] \end{pmatrix}. \qquad (10.6)$$

We first give the tests for p and λ separately. Under the null hypothesis, $(\hat{p}_i-p)/s_{\hat{p}_i}$ is asymptotically normally distributed with mean 0 and standard deviation 1. Hence

$$X_p^2 = \sum_{i=1}^{k} \frac{(\hat{p}_i - p)^2}{s^2_{\hat{p}_i}} \qquad (10.7)$$

is asymptotically distributed as chi-squared with k degrees of freedom. Expression (10.7) could be used to test whether the data are consistent with a given value of p, but usually we are simply interested in testing whether the $\hat{p}_i$ are consistent with each other and we do not know p. In this case we substitute $\hat{p}$ for p and must then reduce the d.f. of (10.7) to k-1. Thus we can test the null hypothesis that all p_i are equal by comparing (10.7), with p set equal to $\hat{p}$, with an upper percentage point of χ^2 with k-1 d.f.

Similarly, we can test the null hypothesis that all λ_i are equal by comparing

$$X_\lambda^2 = \sum_{i=1}^{k} \frac{(\hat{\lambda}_i - \hat{\lambda})^2}{s^2_{\hat{\lambda}_i}} \qquad (10.8)$$

with an upper percentage point of χ^2_{k-1}.

More efficiently, making use of (2.15), we can test the combined null hypothesis that all of the pairs (λ_i, p_i) are equal by comparing

$$X^2_{p,\lambda} = \sum_{i=1}^{k} n_i \left[\frac{(\hat{p}_i - \hat{p})^2 (1-\hat{\lambda})}{\hat{p}\hat{q}(1-2\hat{p}+\hat{\lambda})} - \frac{2(\hat{p}_i - \hat{p})(\hat{\lambda}_i - \hat{\lambda})}{(1-2\hat{p}+\hat{\lambda})} + \frac{\hat{p}(\hat{\lambda}_i - \hat{\lambda})^2}{\hat{\lambda}(1-\hat{\lambda})} \right] \qquad (10.9)$$

with an upper percentage point of χ^2 with 2(k-1) d.f.

Example. We again make use of the Cox and Lewis telephone data (1966). We use the first 60,000 transmissions, arbitrarily dividing them into three successive groups of 20,000 each for convenience even though this fails to illustrate the fact that the group sizes need not be equal. The summary data and parameter estimates of (2.1)-(2.3) are given in Table 5, where s_i is the number of errors in the ith sample, r_i is the number of pairs of adjacent errors, and t_i is the number of errors on the first and last transmissions. It appears that errors occurred more frequently initially and that the subsamples may differ

Table 5. Data Summary and Parameter Estimates for First 60,000 Transmissions Tabulated by Cox and Lewis (1966)

Sample No. i	1	2	3	Pooled
n_i	20,000	20,000	20,000	60,000
r_i	13	0	1	14
s_i	38	0	14	52
t_i	0	0	0	0
$\hat{p}_i$	.001900	0	.000700	.000867
$\hat{\lambda}_i$	.342097	Indeter-minate	.071425	.269228

significantly. We shall test the null hypotheses of no difference between the p_i, between the λ_i, and between the (λ_i, p_i). The tests are not independent of one another.

The estimated variance-covariance matrix (10.6) of each $(\hat{\lambda}_i, \hat{p}_i)$ calculated from the pooled estimates $\hat{p}$ and $\hat{\lambda}$ is

$$\begin{pmatrix} 1.1351 \times 10^{-2} & 1.3450 \times 10^{-5} \\ 1.3450 \times 10^{-5} & 7.5095 \times 10^{-8} \end{pmatrix}.$$

From (10.7) with $p=\hat{p}$,

$$X_p^2 = 24.59 \text{ with 2 d.f.,}$$

whereas even the 0.1% of the chi-squared with 2 d.f., $\chi^2_{2,.001}$, is only 13.82. Hence the $\hat{p}_i$ differ significantly at even the 0.1% significance level.

There are only two $\hat{\lambda}_i$ to compare, but we calculate (10.8) with $\hat{\lambda}$ from all the data still:

$$\chi_\lambda^2 = 3.23 \text{ with 1 d.f., } \chi^2_{1,.05} = 3.84,$$

so the difference between $\hat{\lambda}_1$ and $\hat{\lambda}_3$ is not significant.

Comparing the $(\hat{\lambda}_i, \hat{p}_i)$ pairs presents a non-standard situation since there are three $\hat{p}_i$ and two $\hat{\lambda}_i$. However, it can be argued intuitively that the two terms in $\lambda_2 - \hat{\lambda}$ in (10.9) simply do not exist, or are zero, so that (10.9) becomes

$$X^2_{p,\lambda} = 24.59 + 3.23 - 2.38 = 25.44 \text{ with 3 d.f.,}$$

wheras $\chi^2_{3,.001} = 16.27$. Thus the three samples differ highly significantly, but all of the difference is attributable to the $\hat{p}_i$.

It should be recalled that the tests are approximate because the variance-covariance matrix is estimated from the pooled data. After the $\hat{p}_i$ have been found to differ significantly, it might be felt that further testing should be limited to the two samples that provide estimates of λ_i, even though samples No. 2 and No. 3 differ least in r_i and s_i. Doing this, estimating the variance-covariance matrix from samples No. 1 and No. 3 only, has been carried out and found not to change the high significance of X^2_p and $X^2_{p,\lambda}$ but does change X^2_λ enough to render it significant at the 5% level:

$$X^2_p = 13.78\text{(1 d.f.)}, \; X^2_\lambda = 6.18\text{(1 d.f.)}, \; X^2_{p,\lambda} = 15.32\text{(2 d.f.)}.$$

The conclusion of the statistical analysis is that the telephone transmission process changed more during the first 60,000 transmissions than can be attributed to random sampling fluctuations, and most of the change is attributable to the absolute error rate rather than the degree of dependence between successive transmissions. Some assignable cause for the real change in error rate should be sought.

11. VALIDATION OF INTERVAL FORMULAS BY MONTE CARLO SIMULATION

11.1. Introduction

It is impractical to test the five approximations to a confidence interval for the error rate p derived in Sections 2-6 by comparison with the "exact" interval for the large sample sizes for which they are of most interest. Furthermore, as mentioned in Section 7, the "exact" interval is not really exact

because it, like the approximations, requires knowledge of λ. The theory behind the derivation implies that the interval approximations will cover p with at least the desired frequency $1-2\alpha$ if the sample size is "sufficiently large", but gives no indication what that sample size is in any particular case. While more sophisticated theory might yield information, for the normal approximation in particular, the most readily available method for testing the approximation seems to be simulation of a large number of samples and the corresponding intervals on a large computer.

An extensive program was written for generating samples of Markov chains with given p, λ, and n, calculating all five confidence interval approximations for each sample, and recording the most interesting summary characteristics for 1000 such samples. The parameter values assumed were

λ = .3, .8;

p = .5, .3, .1, .03, .003, .001, .0003.

The sample size n was varied to be at least minimally adequate for estimating p. Thus the (p,n) combination used for both values of λ and α=.05 (90% confidence level) were

p	.5	.3	.1	.03	.03	.003	.001	.0003
n	50	50	100	100	500	1000	1000	1000.

Except for the last two, these were also used for λ=.3 and α=.025. One other case evaluated was λ=.3, α=.025, p=.1, n=500. The smallest p used is nearest to, but still relatively far from, error rates in digital telecommunications; the sample size for it is inadequate (expected number of errors of 0.3 per sample!), but larger sample sizes would lead to larger, perhaps too large, computing costs.

The summary results of all these computations are presented in Appendix A. In general these results consist of the mean value and some measure of dispersion over all 1000 samples of $\hat{p}$, $\hat{\lambda}$, λ^*, the fraction of each type of interval covering p (i.e., the empirical counterpart of $1-2\alpha$), the ratio of the length of the interval to that of the modified Anderson-Burstein

interval, and the number of iterations required for convergence of the three iteratively calculated intervals. In addition, six different events that might occur in a sample and cause difficulty in calculating sensible intervals are counted, as shown at the top of each case summary.

11.2. Simulation Computer Program

Before analyzing the results of the simulations, the computer program for obtaining them will be discussed briefly. The program is very long and is not referenced in this report but may be obtained by writing to Martin J. Miles (Institute for Telecommunication Sciences, U.S. Dept. of Commerce, Boulder, CO 80302).

Using 1000 samples, the program conducts Monte Carlo simulations. Given a value of each of α, p, n, and λ, the program obtains a value of each of s, r, and t (2.3) for each random sample. From (1.1), (1.2), and (1.5), define $p=P[X_i=1]$, $p_{11}=\lambda=P[X_i=1|X_{i-1}=1]$, and $p_{01}=(1-\lambda)p/q=P[X_i=1|X_{i-1}=0]$. Now to obtain s, r, and t, the following procedure was conducted once for each sample:

A random (or, more precisely, pseudorandom) number, y, is generated by the computer[1]. The value of y is such that $0<y<1$. Now if $y\leq p$, an error is indicated, and s and t are incremented by 1 (s, r, and t being initially set at zero). Then n-1 additional random numbers y are generated successively and subjected to one of two tests:

Test 1: If the previous random number indicated an error, and if $y\leq p_{11}$, another error has occurred.

Test 2: If the previous random number indicated an error had not occurred, and if $y\leq p_{01}$, an error has occurred.

If test 1 indicates an error, both s and r are incremented by 1. If test 2 indicates an error, only s is incremented by 1.

---

[1]The random number is obtained by calling the function RANF, which is defined in Fortran Common Library Mathematical Routines, Control Data Corporation.

Finally, if the last random number indicates an error, t is also incremented by 1. If neither test indicates an error, none of s, r, and t is incremented.

With the values of s, r, and t thus determined, the program evaluates and prints the statistics related to the five confidence limit approximations.

The results of the Monte Carlo simulations presented in Appendix A are based on 1000 samples except for two cases of 2000. Further 1000-sample repeat runs were run to establish the biases of $\hat{\lambda}$ and λ^* more precisely but are not included in Appendix A.

11.3. Analysis of the Simulated Point Estimates of p and λ

It is known theoretically that $\hat{p}=s/n$ is an unbiased estimate of p; that is, its expectation $E\hat{p}$ is equal to p whatever p, λ, and n are, and this is confirmed by the simulation summaries in Appendix A in all cases. The mean of the simulated 1000 sample values of $\hat{p}$, to be denoted by $\bar{\hat{p}}$, is of course not exactly p, but it differs only by sampling error as measured by its standard error,

$$s_{\bar{\hat{p}}} = s_{\hat{p}}/(1000)^{\frac{1}{2}},$$

where $s_{\hat{p}}$ is the item labeled SIGMA in the column labeled P HAT of Appendix A. In fact, 21 of the 23 values of $\bar{\hat{p}}$ differ from p by less than $s_{\bar{\hat{p}}}$ (91 percent compared with the theoretical normal 68 percent).

The estimation of λ is known theoretically to be more complicated than that of p, and that is confirmed by the simulations. The estimates $\hat{\lambda}$ and λ^* differ so negligibly from each other that the simpler λ^* seems preferable to $\hat{\lambda}$. In 21 runs of 1000 simulations each for λ=.3 of the samples ranging in size from 50 to 1000 (including some not reproduced in Appendix A), the overall mean $\hat{\lambda}$ was just .00015 less than that of λ^*, ranging from .00155 less to .00014 more. The standard deviation of $\hat{\lambda}$ (in a run of 1000 simulations) averaged just 0.17 percent less

than that of λ^*, ranging from 0.93 percent less to 0.13 percent more. The differences are greater for $\lambda=.8$, but still negligible: In 12 runs of 1000 simulations each the overall mean $\hat{\lambda}$ was .00115 greater than that of λ^*, ranging from .00063 less to .00266 more (more in 11 of the 12 runs). The standard deviation of $\hat{\lambda}$ averaged just 0.05 percent less than that of λ^*, ranging from 3.46 percent less to 0.16 percent more. It should be noted that although there are 1000 simulations in each run, λ cannot be estimated when s=0; hence in a few runs, the samples of $\hat{\lambda}$ and λ^* are relatively small, down to 51 in one case (which has been replaced in Appendix A by a run of 2000 simulations with 111 values of $\hat{\lambda}$ and λ^*).

Although $\hat{\lambda}$ and λ^* differ negligibly from each other, they both differ considerably from λ on the average. Table 6 shows that they have uniformly negative biases. As shown in the table, many of the cases were repeated with different pseudorandom numbers to confirm the results. The standard errors of the mean values of $\hat{\lambda}$ and λ^* (and thus of the biases) based on $s_{\bar{\lambda}}=s_{\hat{\lambda}}/N^{1/2}$ demonstrate the reality of the biases. It was found empirically that the bias is a function of n and p only through the product np, at least approximately. Thus the bias of $\hat{\lambda}$ (or λ^*) is plotted as a function of np in Figure 3 for the two values of λ for which we have information, .3 and .8. Table 6 shows four cases in which the same np is obtained from two different pairs of n and p. In only one of those four cases are the biases significantly different from each other [$\lambda=.8$, $(n,p)=(50,.3)$, $(500,.03)$], and the two biases are shown separately in Figure 3 for that case.

Figure 3 does confirm the theoretical conclusion that the bias of $\hat{\lambda}$ is negligible if the sample size is sufficiently large (Klotz, 1973). However, the bias is not negligible for the sample sizes of many important practical applications. The effect of underestimating λ is to underestimate the length of the confidence interval for p and thus possibly to fail to

Table 6. Monte Carlo Biases of $\hat{\lambda}$ and λ^*

$\lambda = .3$

np	n	p	α	N=no. of estimates	$b(\hat{\lambda})$	$s_{\hat{\lambda}}$	$b(\lambda^*)$	s_{λ^*}
50	500	.1	.025	1000	-.00442	.00210	-.00441	.00210
25	50	.5	.025	1000	-.00761	.00297	-.00764	.00299
			.05	1000	-.01044	.00290	-.01056	.00290
			mean	2000	-.00902	.00207	-.00910	.00208
15	50	.3	.025	1000	-.02174	.00372	-.02129	.00376
			.05	1000	-.01479	.00383	-.01478	.00384
	500	.03	.025	1000	-.02204	.00386	-.02217	.00386
			.05	1000	-.02199	.00386	-.02213	.00386
			mean	4000	-.02012	.00191	-.02009	.00192
10	100	.1	.025	999	-.03204	.00477	-.03206	.00478
			.05	998	-.02956	.00481	-.02948	.00482
				999	-.03234	.00479	-.03247	.00480
			mean	1997	-.03131	.00339	-.03134	.00340
3	100	.03	.025	900	-.08751	.00759	-.08705	.00761
			.05	895	-.08044	.00761	-.07889	.00764
				891	-.09190	.00739	-.09164	.00739
	1000	.003	.025	881	-.07923	.00765	-.07930	.00765
				879	-.07780	.00769	-.07788	.00769
			.05	887	-.07216	.00781	-.07207	.00781
				884	-.07909	.00771	-.07905	.00771
			mean	6217	-.08116	.00288	-.08084	.00289
1	1000	.001	.05	481	-.10006	.01156	-.09979	.01158
.03	1000	.0003	.05	167	-.16780	.01837	-.16763	.01840
				184	-.14687	.01865	-.14668	.01867
				174	-.10934	.02003	-.10910	.02005
			mean	525	-.14134	.01099	-.14114	.01100

Table 6 (continued)

$\lambda = .8$ (all $\alpha = .05$)

np	n	p	N	$\bar{\hat{\lambda}}$	$s_{\hat{\lambda}}$	$\bar{\lambda}^*$	s_{λ^*}
25	50	.5	1000	-.02367	.00288	-.02375	.00298
15	50	.3	979	-.05237	.00539	-.05503	.00550
			995	-.04987	.00478	-.05057	.00492
		mean	1974	-.05112	.00360	-.05280	.00369
	500	.03	954	-.07079	.00616	-.07154	.00617
			959	-.07198	.00617	-.07247	.00616
		mean	1913	-.07138	.00565	-.07200	.00571
10	100	.1	904	-.10444	.00789	-.10635	.00794
			908	-.11363	.00795	-.11585	.00797
		mean	1812	-.10904	.00560	-.11110	.00563
3	100	.03	445	-.15225	.01355	-.15362	.01353
			475	-.18185	.01429	-.18386	.01427
	1000	.003	418	-.15228	.01420	-.15282	.01420
		mean	1338	-.16213	.00809	-.16343	.00809
1	1000	.001	172	-.19751	.02466	-.19926	.02462
0.3	2000	.0003	111	-.24127	.03271	-.24063	.03275

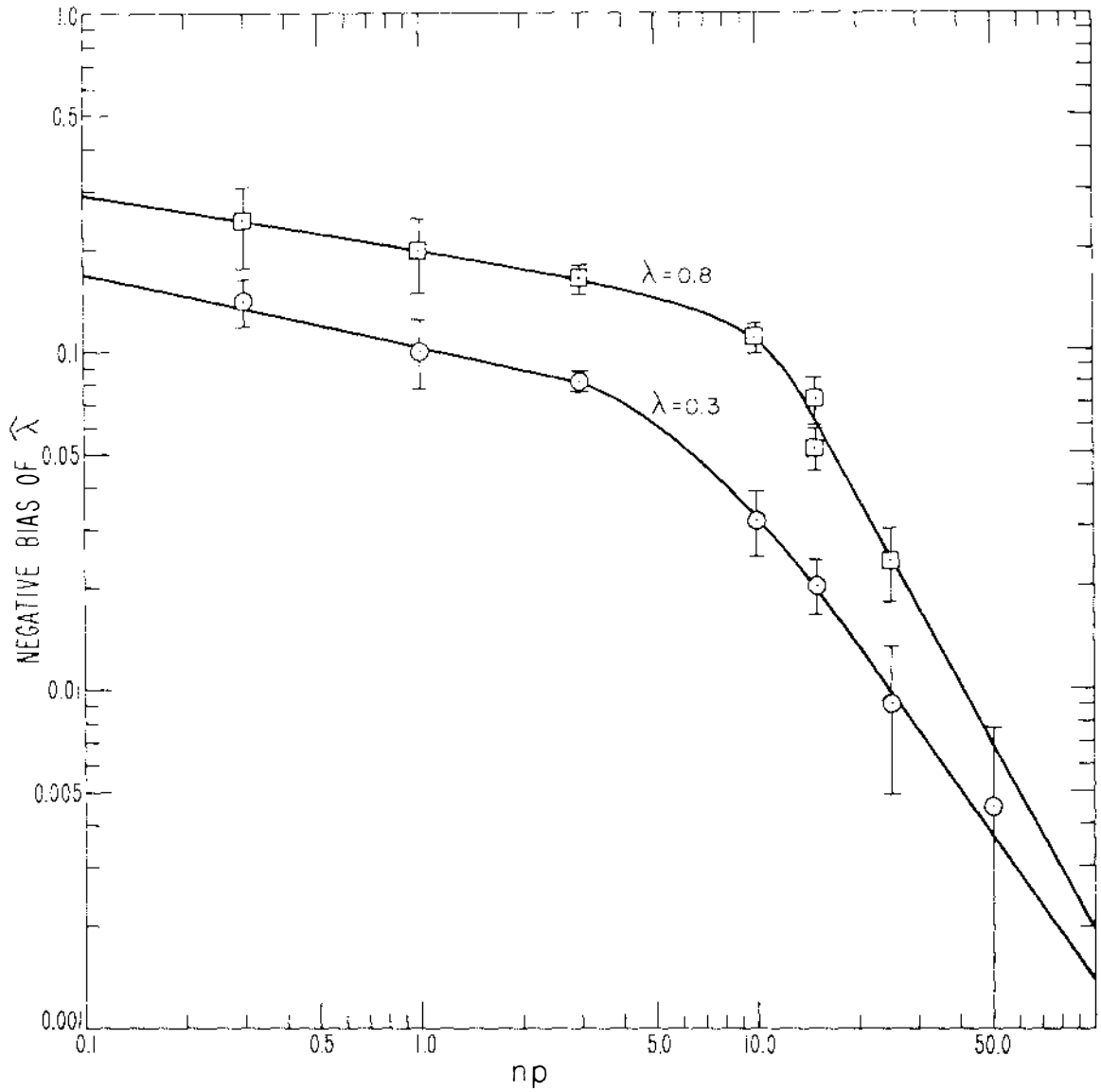

Figure 3. The negative bias of the estimator $\hat{\lambda}$ as a function of np and λ. Approximate 95% confidence (2σ) limits are included.

achieve the stated confidence level. Hence we now introduce a correction, $c_{\hat{\lambda}}$, for the bias of $\hat{\lambda}$ (or λ^*) based on Figure 3.

The bias of $\hat{\lambda}$ is a function of np and λ. We can estimate np unbiasedly by $n\hat{p}=s$, but not of course λ by $\hat{\lambda}$. Nevertheless it is reasonable to seek an approximate removal of the bias with a new estimator $\tilde{\lambda}$ that is a linear function of $\hat{\lambda}$,

$$\tilde{\lambda} = a + d\hat{\lambda}. \tag{11.1}$$

We denote the negative of the bias of $\hat{\lambda}$ for a given λ by c_λ, so that $c_\lambda>0$. The conditional expected (average) value of $\tilde{\lambda}$ for a given s (as well as given n, p, and λ) is

$$E_s\tilde{\lambda} = a + dE_s\hat{\lambda} = a + d(\lambda-c_\lambda). \tag{11.2}$$

We require that $E_s\tilde{\lambda}=\lambda$, and we know c_λ for two values of λ (for each s). Hence,

$$\begin{aligned} .3 &= a + d(.3-c_{.3}), \\ .8 &= a + d(.8-c_{.8}). \end{aligned} \tag{11.3}$$

Solving for a and d and substituting in (11.1) yields

$$\tilde{\lambda} = \frac{\hat{\lambda}+1.6c_{.3}-0.6c_{.8}}{1-2(c_{.8}-c_{.3})}. \tag{11.4}$$

The new estimator $\tilde{\lambda}$ cannot be expected to be exactly unbiased even for given s and $\lambda=.3$ and $\lambda=.8$ because the sample values of $\hat{\lambda}$ vary randomly and the correction applied to a given $\hat{\lambda}$ (not knowing λ, in particular $\lambda=.3$ or $\lambda=.8$) will almost always not be the one appropriate for $\lambda=.3$ or $\lambda=.8$. Furthermore, since $\hat{\lambda}$, $c_{.3}$, and $c_{.8}$ are nonlinear functions of s and the latter two occur nonlinearly in (11.4), the (unconditional) expected value of $\tilde{\lambda}$ could hardly be λ. Nevertheless (11.4) should correct much of the bias of $\hat{\lambda}$ (or λ^*).

The linear fit embodied in (11.4) may be expected to interpolate better than it extrapolates. In extreme cases it even produces values of $\tilde{\lambda}$ outside the admissible interval (0,1). For example, when s=10 and r=0, $\lambda^*=0$, and $\tilde{\lambda}=-.016$ from (11.4) and Figure 3, and when s=10, r=9, and n=100, $\lambda^*=.909$, and $\tilde{\lambda}=1.058$. [Here (11.4) is applied substituting the simpler λ^* for $\hat{\lambda}$.] Hence it is desirable to use two other interpolation formulas (similarly derived) for $\hat{\lambda}<.3-c_{.3}$ and $\hat{\lambda}>.8-c_{.8}$:

$$\tilde{\lambda} = \frac{0.3\hat{\lambda}}{0.3-c_{.3}}, \qquad 0 \le \hat{\lambda} \le 0.3-c_{.3}; \tag{11.5}$$

$$\tilde{\lambda} = \frac{0.2\hat{\lambda}+c_{.8}}{0.2+c_{.8}}, \qquad 0.8-c_{.8} \le \hat{\lambda} \le 1. \tag{11.6}$$

At their common points of applicability, (11.5) and (11.6) yield the same values (0.3 and 0.8 respectively) of $\tilde{\lambda}$ as (11.4), which then is used only in the interior interval, $0.3-c_{.3} \le \hat{\lambda} \le 0.8-c_{.8}$. It can be easily shown from (11.4)-(11.6) that $\tilde{\lambda} \ge \hat{\lambda}$.

When λ is estimated to be 1, all of the approximate confidence intervals for p blow up. In effect there is only one independent observation, and the confidence interval for p is consequently to be taken immediately as ($p_L=0$, $p_U=1-\alpha$) if s=0 and ($p_L=\alpha$, $p_U=1$) if s=n. However, λ is rarely, if ever, estimated as 1 from (11.6).

To assess the estimator $\tilde{\lambda}$, six of the simulation cases were repeated (one of them with 2000 samples) with the same random numbers, but with $\tilde{\lambda}$ calculated from (11.4-11.6) and substituted for $\hat{\lambda}$ in determining confidence intervals. The summaries of these runs are the last six in Appendix A and are easily identified by the additional column of MEAN, MIN, MAX, and SIGMA labeled LAMBDA TILDE. The estimation is further summarized as follows:

λ	p	n	N	$\bar{\hat{\lambda}}$	$\bar{\tilde{\lambda}}$	$s_{\hat{\lambda}}$	$s_{\tilde{\lambda}}$
.3	.1	100	999	.268	.301	.152	.169
	.0003	1000	174	.191	.241	.264	.331
.8	.3	50	995	.750	.805	.151	.136
	.1	100	913	.684	.752	.240	.243
	.003	1000	418	.648	.721	.290	.302
	.0003	1000	111	.559	.631	.345	.371

The column labeled N is the number of samples obtained less the number of samples for which s=0 (and thus for which λ cannot be estimated). We see that $\tilde{\lambda}$ is indeed less biased than $\hat{\lambda}$; most, even all, of the bias is removed if np is sufficiently large,

greater than 10 say, and $\hat{\lambda}$ is not too near 1. The reduction in bias is achieved at the expense of a slight increase in variance in 5 of the 6 cases.

Hence it has been confirmed empirically that the empirical modification of $\hat{\lambda}$ (or λ^*) to $\tilde{\lambda}$ does remove much of the bias in estimating λ. It remains to be seen in Section 11.4 below what the effect on the confidence limits is.

11.4. Analysis of the Simulated Confidence Intervals for p

The computer printout summaries of Appendix A contain considerable information on the distributions of confidence intervals of simulated samples. The essence of it is most easily grasped from Figures 4-6, in which the fraction of the intervals covering p is plotted vertically and the average ratio of the length to the length of the modified Anderson-Burstein interval for the same sample is plotted horizontally. Figures 4 and 5 give the results for λ=.3, 90% and 95% confidence level, respectively, while figure 6 gives the results for λ=.8 and 90% confidence level. All of these intervals were calculated using $\hat{\lambda}$, since the bias of $\hat{\lambda}$ and λ^* was discovered from these same simulations. The effect of substituting $\tilde{\lambda}$ will be discussed later.

Since the measures of dispersion in Appendix B indicate that both the fractions of coverage and the average relative lengths are correct to about the nearest .01, we can immediately draw the following conclusions from Figures 4-6 about the five approximate confidence interval formulas for p calculated using $\hat{\lambda}$ to estimate λ:

1. When λ=.3 and np (the expected number of errors or "successes") is greater than about 3, all five formulas satisfy the specification that the fraction of intervals covering p is at least as large as the confidence level. (This holds for the two confidence levels experimented with, 90% and 95%, and past experience suggests it will hold more

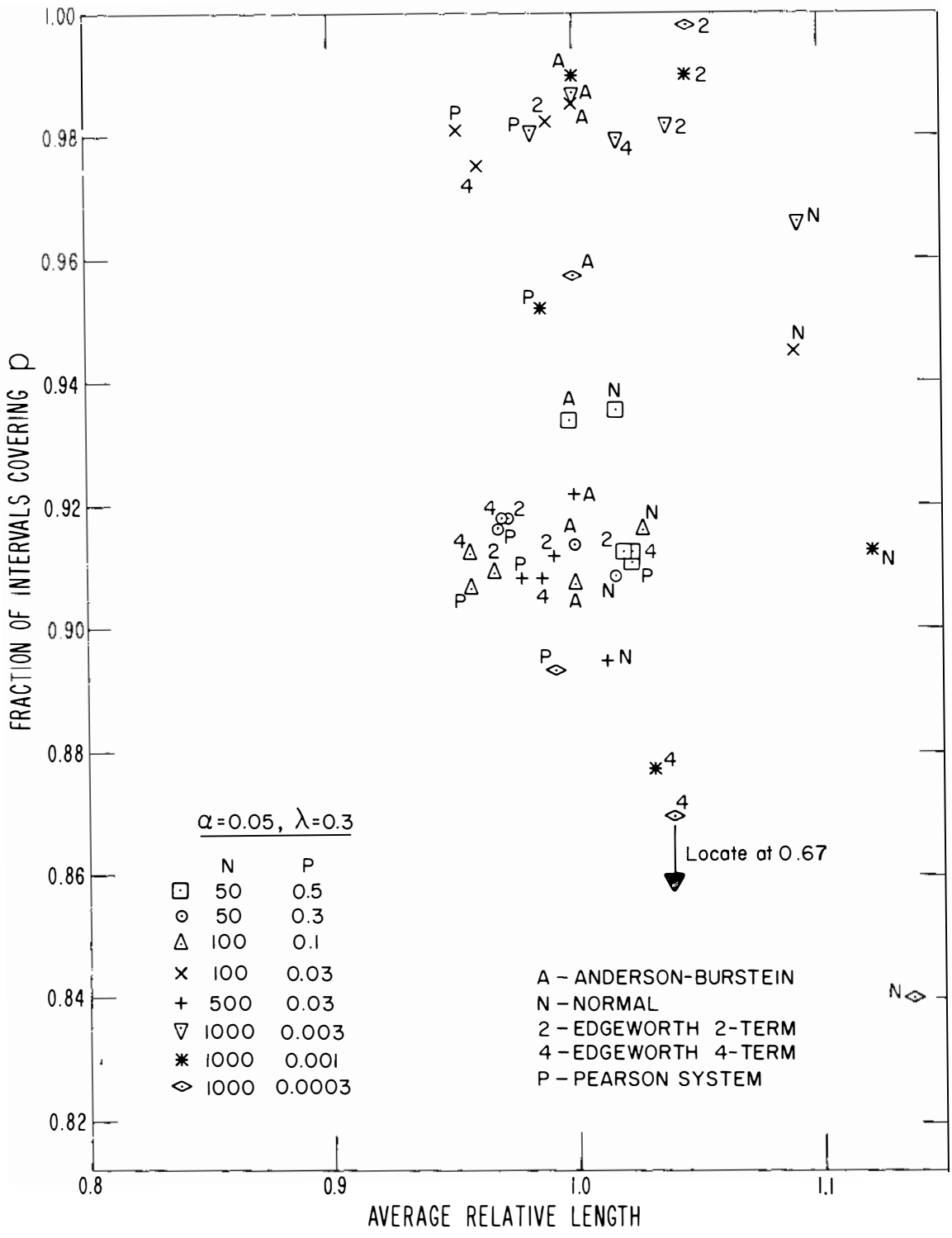

Figure 4. Plot of fraction of 90% confidence intervals for p from 1000 simulations with λ=.3 that actually cover p (empirical confidence level) versus the average of their relative lengths.

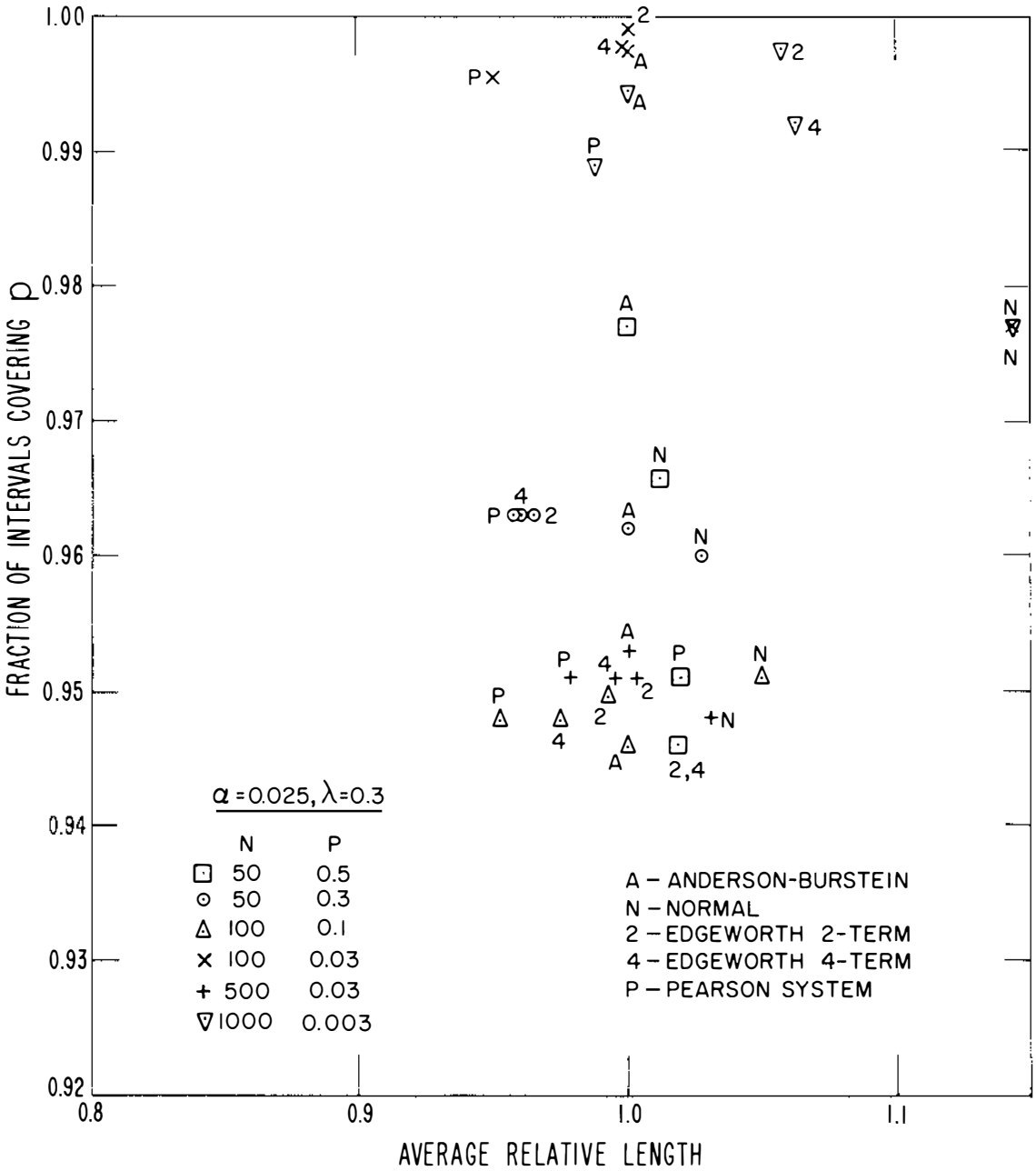

Figure 5. Plot of fraction of 95% confidence intervals for p from 1000 simulations with λ=.3 that actually cover p (empirical confidence level) versus the average of their relative lengths.

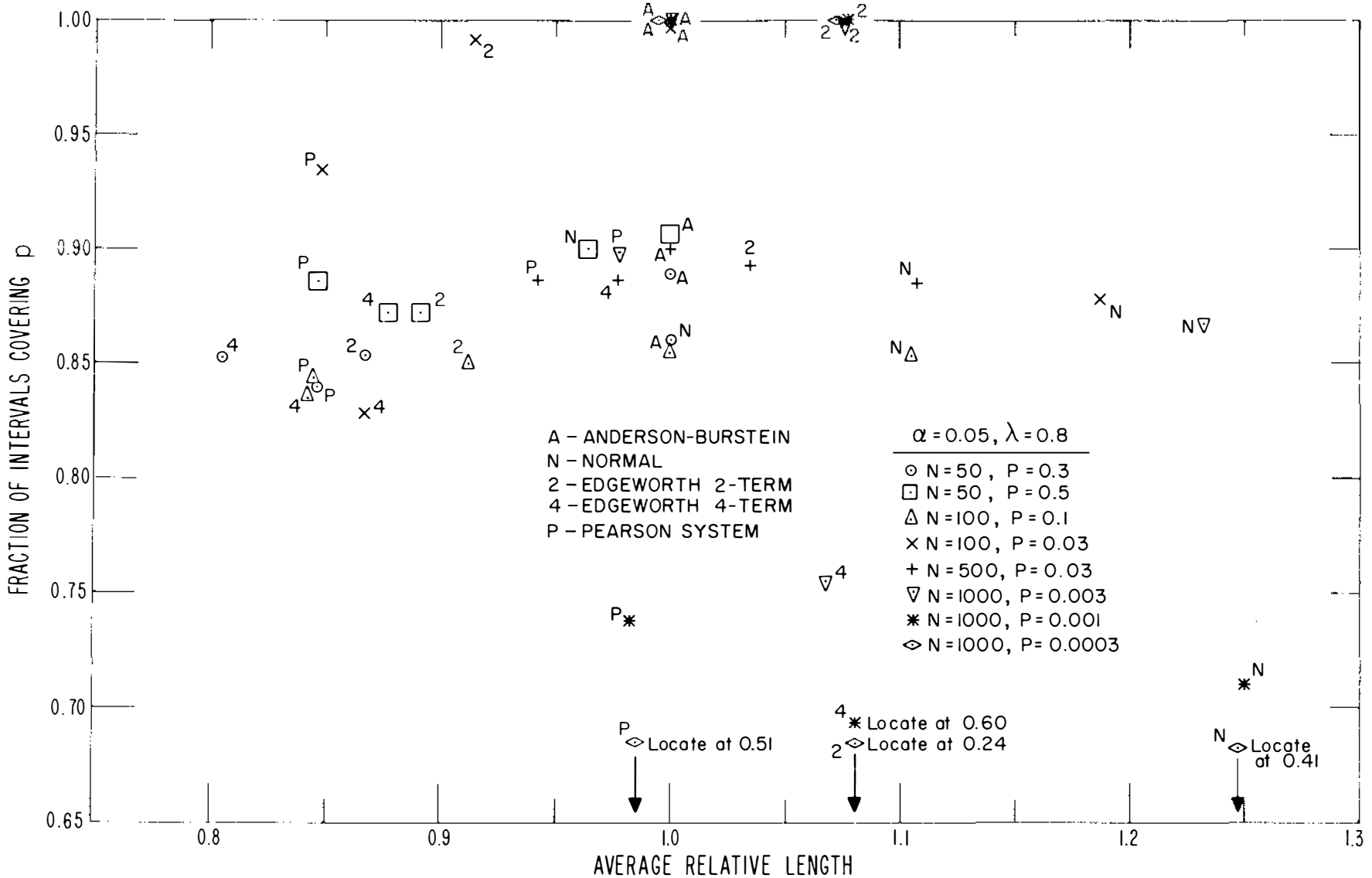

Figure 6. Plot of fraction of 90% confidence intervals from 1000 or 2000 simulations with $\lambda=.8$ that actually cover p (empirical confidence level) versus the average of their relative lengths.

strongly with smaller levels, like 80%, less strongly with larger levels, like 99%.) As np decreases below about 1, the coverage becomes more erratic.

2. When λ=.8, the fraction of intervals covering p fails to attain the specified confidence level (90%) in 27 of the 40 cases plotted, although it falls greatly below (<83%) in only 7 cases, all of which have $p \leq .003$ and $np \leq 3$.
3. There is no correlation between fraction of coverage and average relative length of the various types of intervals. The normal interval tends to be longest and the Pearson the shortest, but the Pearson has just as good coverage.
4. The Pearson, Edgeworth 2-term, and Edgeworth 4-term approximations, which are the intervals requiring iterative solution, tend to have the same coverage and length.
5. The modified Anderson-Burstein interval, which is the easiest to calculate, tends to have better coverage than the other intervals, failing to cover the specified fraction of the time only by small amounts in just 2 cases. It is intermediate in length to the normal interval and the three iterative solutions, although the differences in length are not large. The good behavior of the modified Anderson-Burstein interval is not surprising in our many cases of small p, since it is a generalization of the Poisson approximation for independent trials, but the good behavior for large values of p also is a pleasant surprise.

Most of the parameter combinations were simulated 1000 times, but for small np many cases of s=0 occur, for which λ cannot be estimated. The most extreme case is that of p=.0003, λ=.8, n=1000, for which only 51 values of $\hat{\lambda}$ and λ^* could be calculated. Subsequently the number of simulations for this case was doubled, resulting in 111 samples with s>0, as shown in Appendix A. Further simulations were obtained in some other cases also, as recorded in Table 6, up to 3000 in one case, but not all are reproduced in Appendix A.

We turn now to the effect on the confidence interval approximations of replacing $\hat{\lambda}$ (or λ^*) by $\tilde{\lambda}$. Since $\tilde{\lambda} \geq \hat{\lambda}$, it seems intuitively that an interval based on $\tilde{\lambda}$ should be larger than (and cover) the corresponding interval from the same sample based on $\hat{\lambda}$ and hence cover p more often. From (6.1) and (6.2) it is seen immediately that this is true for the modified Anderson-Burstein interval. Since Var s in (2.6) increases with ρ, it can be seen from (2.8) that $\hat{V}$ increases with $\hat{\rho}$ (at least for large n) and from (2.13) that the normal approximation's upper limit p_{U0} increases with $\hat{\rho}$. By taking the derivative, it also follows that p_{L0} decreases as $\hat{\rho}$ increases. The Edgeworth and Pearson limits are too complicated for such simple analysis, but, as refinements of the normal limits, are expected also to expand if $\hat{\lambda}$ is replaced by the larger $\tilde{\lambda}$. Hence the use of $\tilde{\lambda}$ should yield a larger fraction of coverage of p than the use of $\hat{\lambda}$, and the change should be greater when the bias of $\hat{\lambda}$ is greater, and that occurs when np is smaller, less than about 10. The cases of inadequate coverage in Figures 4 and 6 tended to be those for which $np \leq 3$. Ideally a system of confidence intervals should be just long enough to provide the specified probability of coverage and no longer. The use of $\tilde{\lambda}$ appears to help meet that criterion.

To verify and quantify the improvement in confidence intervals by the use of $\tilde{\lambda}$, six simulation runs were made with 90% intervals calculated using $\tilde{\lambda}$ rather than $\hat{\lambda}$ on the identical samples. These cases form the last six pages of Appendix A and are further summarized in Table 7. The following conclusions can be drawn.

6. The fraction of intervals using $\tilde{\lambda}$ that cover p is never less than that of intervals using $\hat{\lambda}$, but the increase is small for intervals already covering with the specified confidence level of 0.90 and usually dramatically large for those with poor coverage: from .243 to .919, .405 to .667, .514 to .667, and .586 to .891. There was one exception, from .775 to only .813. Failure to achieve the specified coverage is associated with failure of $\tilde{\lambda}$ to

Table 7. Comparison of Characteristics of 90% Confidence Intervals for p Calculated Using $\hat{\lambda}$ and $\tilde{\lambda}$

λ	p	n	Type of interval	Fraction of intervals covering p		Average relative length
				$\hat{\lambda}$	$\tilde{\lambda}$	
.3	.1	100	And.-Burstein, A	.907	.913	1.050
			Normal, N	.916	.920	1.051
			Edgeworth 2	.909	.916	1.046
			Edgeworth 4	.912	.916	1.043
			Pearson, P	.907	.914	1.044
.3	.0003	1000	A	.937	.937	1.144
			N	.805	.885	1.196
			2	.994	1.000	1.171
			4	.586	.891	1.194
			P	.874	.885	1.158
.8	.3	50	A	.880	.920	1.170
			N	.861	.895	1.187
			2	.858	.904	1.148
			4	.858	.904	1.136
			P	.841	.887	1.118
.8	.1	100	A	.843	.909	1.226
			N	.844	.860	1.308
			2	.835	.903	1.198
			4	.818	.888	1.207
			P	.826	.889	1.169
.8	.003	1000	A	1.000	1.000	1.238
			N	.866	.895	1.375
			2	.998	.998	1.302
			4	.775	.813	1.356
			P	.900	.923	1.283
.8	.0003	1000	A	1.000	1.000	1.238
			N	.405	.667	1.379
			2	1.000	1.000	1.313
			4	.243	.919	1.368
			P	.514	.667	1.288

remove all of the bias in $\hat{\lambda}$, as shown in the text tabulation in section 11.3.

7. As expected, the $\tilde{\lambda}$ intervals are longer than the $\hat{\lambda}$ intervals. (The "average relative length" in Table 7 is not the average of the ratios of the $\tilde{\lambda}$ interval length to the $\hat{\lambda}$ interval length in the same sample but is an indirect measure of their relative lengths obtained by taking a ratio of the ratios available in the Appendix A summaries.) The increase in length is small when no increase in length is needed, e.g., in the first case, λ=.3, p=.1, and n=100.

Thus the estimator $\tilde{\lambda}$ is successful in improving the performance of the confidence interval approximations, but it is not perfect because it does not remove the bias in $\hat{\lambda}$ completely.

The last type of information on the simulation summaries is the number of iterations required for the iterative intervals (Edgeworth 2- and 4-term and Pearson) to reach agreement of the last two iterates, both upper and lower limits, to either 3 significant figures or 4 decimal places, whichever is attained first. The mean, minimum, and maximum number of iterations is given for each type of interval.

If such agreement was not reached in 20 iterations, the computer was instructed to stop and accept the last interval. (It is simply assumed that agreement was good enough then.) The last line of the summary, labeled "No. of Failures," gives the number of samples for which agreement was not reached. Nineteen of the 31 summaries (of 1000 simulations each except for two with 2000 simulations) show no failures at all, and there are only 4 summaries that show more than 30 failures:

Table	α	n	p	λ	No. of Failures		
					Edge. 2	4	Pearson
A.11	.025	50	.5	.3	60	77	0
A.18	.05	50	.5	.8	307	298	399
A.19	.05	50	.3	.8	172	185	332
A.28	.05	50	.3	.8	240	272	345

The first three of these apply to intervals calculated from $\hat{\lambda}$, the last one to intervals calculated from $\tilde{\lambda}$. A complete explanation of the incidence of the failures to converge is lacking, but apparently small sample size and large λ are factors, not offset by the zero or small skewness of the distribution of s. The Pearson system iterations converge better than the other two for λ=.3 but worse for λ=.8. Even in the above four summaries agreement between successive iterations was often reached fairly early; some of the simulations required as few as 2 iterations, as can be seen at the bottom of Tables A.18, A.19, and A.20.

Despite the above concern with failures to converge, it is emphasized that the intervals can all be calculated and agree well for practical needs in the great majority of cases. The five intervals produced by the user program of Section 8 can be inspected and any appreciable lack of agreement noted. Often, perhaps usually, there will be no need to go beyond the modified Anderson-Burstein and normal approximations.

12. DESIGNING THE EXPERIMENT

12.1. Introduction

The design of an experiment to estimate an error rate, p, when trials are dependent is necessarily more complicated than when they are independent, but the goals enunciated in OT Report 74-51 (Crow, 1974) apply: to estimate p with a specified

precision or a specified budget, and both if possible. The following discussion will be restricted to small p, say $p<0.1$. Bai (1975) has generalized DeGroot's (1959) result for independent trials that the only sampling plans that yield efficient estimates of p are the plans with prescribed sample size and the plans with prescribed number of errors. Bai's results are complex and are concerned only with transition probabilities, and will not be specifically applied here. However, combined with DeGroot's result, they suggest that to estimate both p and λ efficiently one should sample until both a prescribed number, say c, of errors and a prescribed number, say c', of pairs of successive errors are attained. If one is not interested directly in λ, then it is not essential to prescribe the number of pairs as c', but it is clear from the confidence limit formulas for p that λ must be estimated at least minimally if p is to be estimated reasonably well.

If λ is near p, then there is little difference from independent trials and it will not be important to determine λ itself with precision. On the other hand, if λ is large, then obtaining a sufficient number, c, of errors will also secure, with high probability, a number of pairs of errors of the same order as c (but necessarily less than c). The approach taken here will be to prescribe the number of errors as c both to estimate p with a prescribed relative precision and to estimate λ with a prescribed absolute precision. The sample size, n, will then not be prescribed and may exceed that permitted by a given budget. The sampling may be terminated at the sample size permitted by the budget, or allowed to proceed until c errors are attained, and confidence limits calculated by the formulas of Sections 2 through 6. These formulas are all large sample size approximations and thus apply whether n is prescribed or not. If c errors have not been attained, then the prescribed precision will of course not be attained either.

12.2. Estimating p with a Prescribed Precision

The case in which there is no direct interest in estimating λ with prescribed precision is considered first. It is necessary to be given the maximum value of λ expected, λ_{max}. First c is determined so as to estimate $p \leq 0.1$ with prescribed relative precision as if the trials are independent. This is done from Crow's Figure 3 [1974, Example (2), page 36], thus obtaining a value of c that we shall denote c_{ind}. From (2.1) and (2.2)

$$\sigma^2_{\hat{p}} \doteq \frac{pq}{n} \cdot \frac{1+\rho}{1-\rho} = \frac{pq}{n} \cdot \frac{1-2p+\lambda}{1-\lambda} . \tag{12.1}$$

That is, the variance of $\hat{p}$ in the dependent case is the variance of $\hat{p}$ for the independent case multiplied by

$$\frac{1-2p+\lambda}{1-\lambda} \leq \frac{1+\lambda_{max}}{1-\lambda_{max}} . \tag{12.2}$$

Hence to achieve a prescribed precision (measured by $\sigma_{\hat{p}}$) approximately, it is necessary to multiply the sample size, n, by $(1+\lambda_{max})/(1-\lambda_{max})$. For the large sample sizes to which (12.1) applies, there is no difference in the precision measures such as $\sigma_{\hat{p}}$ or confidence limits for direct sampling (prescribed n) and inverse sampling (prescribed number of errors c). Hence the number of errors to be observed in inverse sampling may be multiplied by the same factor. Thus, to estimate p with a prescribed relative precision, the sampling should continue until the number of errors (or "successes") is

$$c = c_{ind} \frac{1+\lambda_{max}}{1-\lambda_{max}} . \tag{12.3}$$

Example 1. It is desired to estimate the bit error probability p of a telecommunications system with 90% confidence limits within about 50% of p. While p is known to be small, surely less than 0.1, the conditional probability, λ, of an error, given that an error has just occurred, is surely much larger but known to be less than 0.5. How should the experiment be done?

Solution. To attain the prescribed relative precision of ±50%, a stream of bits should be transmitted until a prescribed number of errors c has been observed. To determine c, we first read from Figure 3 of OT Report 74-51 (Crow, 1974), using the curve labeled 90%, the number c_{ind}=12, the number of errors if trials were independent. Since λ_{max}=0.5, we calculate

$$c = c_{ind} \frac{1+\lambda_{max}}{1-\lambda_{max}} = 12 \cdot \frac{1.5}{0.5} = 36.$$

Hence we should run the experiment until 36 errors have been observed.

Example 2. In Example 1 the experimenter is told that the budget will not permit a sample size any larger than 20,000. How does this affect the sampling plan and the achievable precision?

Solution. Sampling should proceed until 36 errors have been observed or until 20,000 bits have been transmitted, whichever occurs first. The precision attained will be the prescribed precision or better if sampling continues until 36 errors have occurred and the sample estimate of $\hat{\lambda}$ turns out to be less than the guaranteed limit λ_{max}=0.5 on λ. Even if λ is less than λ_{max}, $\hat{\lambda}$ may be greater than λ_{max} and result in a precision calculated from the sample (as measured by confidence limits) short of that prescribed, but that should not occur often if λ_{max} is correctly prescribed. If sampling is stopped at n=20,000 characters with less than 36 errors observed, then the error probability will be estimated by $\hat{p}$=s/20,000, s=0,1,...,35, with bounds on 90% confidence limits that can be predicted by the Poisson-Anderson-Burstein method. The factor

$$\left(\frac{1+\hat{\rho}}{1-\hat{\rho}}\right)^{1/2}$$

in (6.1) and (6.2) is replaced by its upper bound

$$\left(\frac{1+\lambda_{max}}{1-\lambda_{max}}\right)^{1/2}$$

which has the value $\sqrt{3}$=1.732 in this example. If s were to turn out to be 35, then from Table 1 of OT Report 74-51, L=25.9, U=46,

$$\hat{p} = .00175,\ p_{LI} = .0013,\ p_{UI} = .0023;$$
$$p_U = .00175 + (.0023-.00175)1.732 = .00270,$$
$$p_L = .00175 - (.00175-.0013)1.732 = .00097.$$

The half-length of the 90% confidence interval is then .00086, which is 49% of the estimate of p, .00175, and thus still happens to satisfy the desired 50% relative precision even though the required number of errors for that, 36, has not been attained. That simply results from the fact that the determination of c is graphical and thus hardly accurate to two digits.

As s decreases (with n fixed at 20,000), the half-length of the confidence interval decreases, so that the absolute precision improves, but the relative precision worsens:

s	$\hat{p}$	p_U	p_L	p_U-p_L	$50(p_U-p_L)/\hat{p}$
35	.00175	.00270	.00097	.00173	49%
30	.00150	.00245	.00077	.00168	56%
20	.00100	.00179	.00042	.00137	68%
10	.00050	.00111	.00010	.00101	101%
5	.00025	.00074	.00000	.00074	148%
0	.00000	.00026	.00000	.00026	∞ .

These predicted confidence limits are all calculated as illustrated for s=35 with the maximum "expansion factor" $\sqrt{3}$ expected. Calculations like these indicate what can be expected from a proposed experiment design and enable a suitable compromise between precision and expense to be determined. After the experiment is performed, the data can be used to calculate $\hat{\lambda}$ as well as $\hat{p}$ and one or more approximations to the confidence limits.

12.3. Estimating λ With a Prescribed Precision

The conditional probability of an error given an error on the previous trial, λ, can be estimated with at least a specified precision in large samples without knowing an upper bound λ_{max} because pairs of errors behave effectively independently in the first-order Markov model that has been assumed throughout this report. This follows from Klotz's (1973, Sec. 4) demonstration that $\hat{\lambda}$ is asymptotically normally distributed with asymptotic variance

$$\sigma^2_{\hat{\lambda}} = \frac{\lambda(1-\lambda)}{np} . \tag{12.4}$$

Equation (12.4) was derived from the likelihood function of the sample, and that function is the same whether the sampling is direct or inverse. In direct sampling, np is the expected number of errors (successes), E(s); in inverse sampling, the number of errors, s, is prescribed as c, and c=pE(n). Asymptotically in either case, np in (12.4) can therefore be replaced by the observed number of errors, s or c, so that

$$\sigma^2_{\hat{\lambda}} = \frac{\lambda(1-\lambda)}{c} , \tag{12.5}$$

which is the same as the formula for c independent trials in which λ is the probability of a success ("success" being a pair of successive errors in this case). The right-hand side of (12.5) increases steadily as λ tends toward 1/2 from either side but changes little for $0.3 \le \lambda \le 0.7$. Hence if λ has any possibility of being in that range, we may as well replace (12.5), for design purposes, by

$$\mathrm{Max}\,\sigma^2_{\hat{\lambda}} = \frac{1}{4c} . \tag{12.6}$$

If λ is not expected to be in the range 0.3 to 0.7, then to avoid planning for too large an experiment [by the use of (12.6)] it is necessary to use the maximum expected value of (12.5), which is obtained by replacing λ by λ_{max} (necessarily specified) for $\lambda \le \lambda_{max} < 0.3$ or by replacing λ by λ_{min} for $\lambda \ge \lambda_{min} > 0.7$.

Equation (12.6) [or (12.5) similarly] is used in experiment design by specifying the max $\hat{\sigma}_{\hat{\lambda}}$ acceptable and solving for c. Alternatively $2\hat{\sigma}_{\hat{\lambda}}$ can be taken in large samples as the half-length of the 95% confidence interval for λ and thus that length can be prescribed to determine c. Another value for c may have been determined to estimate p with prescribed relative precision as in Section 12.2, which requires a specification of λ_{max}. Then inverse sampling would be carried out until the number of errors reaches the larger of these two values of c (unless the sample size first reaches the limit controlled by the budget).

Example 3. In Example 1 it is also desired to estimate λ with 90% confidence limits expected to be within 0.1 of the estimated $\hat{\lambda}$. Should the experiment design be modified?

Solution. The maximum value of $\hat{\sigma}_{\hat{\lambda}}$ to be allowed is 0.1/1.645=0.0608. Hence

$$c = \frac{1}{4\text{Max}\hat{\sigma}_{\hat{\lambda}}^2} = \frac{1}{4\text{x}.00370} = 68.$$

The experiment should be continued until 68 errors (rather than 36 as in Example 1) are observed. If a bound on the sample size of 20,000 has also been imposed as in Example 2, that may override the goal of getting 68 errors.

Example 4. Suppose in Example 3 that the experiment has been carried out to 68 errors and results in a sample size n=21,300, s=c=68 errors, r=43 pairs of errors, and first and last observations not errors so that t=0 in the notation of Section 2. Calculate $\hat{p}$, $\hat{\lambda}$, and 90% confidence limits for p and for λ.

Solution.

From (2.1) $\hat{p} = 68/21,300 = .0031925.$

From (2.2)

$$\hat{\lambda} = \frac{1}{2\text{x}67.9968}\,[-25+135\text{x}.0031925+(603.636+172\text{x}67.9968\text{x}.993615)^{\frac{1}{2}}]$$

$$= .63235.$$

This differs little from the relative frequency estimate (2.4),

$$\lambda^* = 43/(68-.00319) = .63238.$$

From (6.1)

$$\hat{\rho} = .62916/.99681 = .63117,$$

$$\left(\frac{1+\hat{\rho}}{1-\hat{\rho}}\right)^{\frac{1}{2}} = 2.1030,$$

$$p_U = .00319 + (83/21,300-.00319)2.1030 = .00467,$$

$$p_L = .00319 - (.00319-55/21,300)2.1030 = .00191.$$

From (2.8) and (2.13)

$$\hat{V} = \frac{21,300x.99681}{(.36883)^2}\,(1-.6117^2-1.26234/21,300) = 93,891,$$

$$p_{U0} = (2x21,300^2)^{-1}(93,891x1.6449^2+137x21,300+1,243,810) = .00487,$$

$$p_{L0} = (2x21,300^2)^{-1}(93,891x1.6449^2+135x21,300-1,235,082) = .00209.$$

Since these normal approximation limits are not far from the preceding Poisson-Anderson-Burstein limits, it is not essential to calculate the more complicated Edgeworth or Pearson approximation limits. We can assert $.0020 \le p \le .0048$ with 90% confidence. The relative precision in estimating p is

$$\frac{.00487-.00209}{2x.00319} = .435 \text{ or } 43.5\%,$$

better than the 50% relative precision sought in Example 1 since the experiment was run to 68 errors rather than 36, although the improvement is partly canceled by the fact that $\hat{\lambda}$ turned out somewhat larger than the λ_{max} assured prior to the experiment, 0.5.

Approximate 90% confidence limits for λ are, by (2.18),

$$\hat{\lambda} \pm 1.6449[\hat{\lambda}(1-\hat{\lambda})/c]^{\frac{1}{2}} = .63235 \pm .09618$$

$$= .536 \text{ and } .729.$$

Thus the goal of estimating λ with absolute precision ± 0.1 (as measured by 90% confidence limits) has been attained.

If the experiment had been truncated at n=20,000, then the desired precision for p and λ might not have been attained, but confidence limits for them can be calculated in the same way.

13. CONCLUSIONS AND RECOMMENDATIONS

1. Confidence limits for the digital error rate p of a communication system for which successive transmissions are dependent may in principle be calculated by Ladd's (1975) algorithm for the exact cumulative probability of the number of errors or from any one of the five approximate formulas derived herein. Ladd's algorithm appears to be practical only for smaller sample sizes (up to the order of 100 or 200) and only if a large computer is available. The approximate formulas are satisfactory for sample size n as small as 50 if the number of errors (successes) s is 10 or more; they are useable down to s=2, and they become better approximations as n increases. It is necessary either to know λ, the conditional probability of an error given an error on the preceding transmission or to estimate λ from the data using (2.4) and (11.4)-(11.6). All of the results apply also to the general statistical problem of calculating confidence limits for a proportion from dependent trials.
2. The five approximate confidence intervals are, in order of ease of calculation, the modified Poisson-Anderson-Burstein (Anderson-Burstein more briefly) (6.1)-(6.2), the normal (2.13), (2.8), the Edgeworth two-term (3.11), (3.7), (3.4), the Edgeworth four-term (4.4), (4.3), (3.4), (3.11), and the Pearson system (5.1), (4.3), (3.4), (3.11). In all cases, if $s \geq 2$, λ is estimated by (11.4)-(11.6) using either $\hat{\lambda}$ (2.2) or λ^* (2.4) to get $\tilde{\lambda}$. If s=0 or s=1, a prior or external estimate of λ should be used; furthermore the lower confidence limit should be taken as 0 and the upper confidence limit should be calculated by the "exact" formulas (7.3) or (7.4), respectively. All five approxima-

tions may be obtained simultaneously using the computer program of Section 8 and Appendix B.

3. The extensive set of simulations of Section 11 shows that, for the large range of sample sizes used, the two approximations making use of the fourth moment of the number of errors (successes), the Edgeworth four-term and the Pearson system, provide no advantage over the Edgeworth two-term, which uses only the first three moments. They are retained in the computer program of Section 8 for comparison, however, and for sufficiently large sample sizes may well provide better approximations.

4. Use of the computer program of Section 8 is not essential but is recommended if a substantial number of samples are to be analyzed. The Anderson-Burstein and normal approximations are calculated easily in a few minutes on a pocket calculator. It is recommended that they be obtained first. If they agree closely enough (i.e., to the user's satisfaction), then no further calculation is needed. The normal approximation usually is the poorest of the five approximations. The Edgeworth two-term (as well as the four-term and Pearson system) may require some iteration to verify the solution and thus may be somewhat tedious without a computer program, but is useful if there is any question of the closeness of the normal approximation. For small p, say ≤ 0.1, the lower confidence limit tends to be a poorer approximation relatively than the upper limit (for any of the five formulas), but the user may well be interested only in an upper limit anyway. If so, his confidence level is $1-\alpha$ (e.g., 95% rather than 90%).

5. Klotz's (1973) estimator $\hat{\lambda}$ of λ and the estimator λ^* proposed herein (2.4) [as well as Devore's (1976) modified maximum likelihood estimator apparently] are substantially biased when the expected number of errors (successes) is small, say less than 10 (Section 11.3 and Figure 3). Much of the bias is removed by the empirical transformation (11.4)-(11.6) to $\tilde{\lambda}$.

6. The validity of the stationary first-order Markov chain model may be tested by the methods of Section 10. In practice, non-stationarity due to change of conditions may be the most common violation of the assumptions and should be tested and dealt with as in Section 10.4. Even if the system is not strictly a first-order Markov chain, that model can be considered a first-order approximation to the true model of dependence and will very often be a satisfactory approximation. The change in going from it to the true model would usually seem to be less than the change in going to it from the independence (Bernoulli or binomial) model, so the latter change can be used as a basis for judgment.
7. An experiment for estimating the error rate p should be designed with care, as in Section 12. A prior estimate of (or upper bound on) the conditional error rate λ is required. The approach is to determine the number of errors (successes) to be observed in order to estimate p with a prescribed relative precision and to estimate λ with a prescribed absolute precision. Example 1 of Section 12 shows that the information on p will be very minimal unless the number of errors observed is of the order of 10 or more.
8. Full information on the first-order Markov model is gained by a normal approximation joint confidence region for (λ,p) as specified by equation (2.16) and illustrated in Section 9 and Figure 2. This region is included in the computer program of Appendix B.

14. ACKNOWLEDGMENTS

The work described in this report was sponsored by the Office of Telecommunications, U.S. Department of Commerce, as part of the computer communications project. The authors are indebted to Dr. Peter McManamon, the group chief and project

leader (during most of the work), for suggesting the topic and encouraging the work's progress; to Patricia A. Moreno for typing the report; to John Harman and Felicitas Whelan for drafting the figures; to Dr. John F. LaBrecque of the National Bureau of Standards and to Dr. Arthur D. Spaulding of the Office of Telecommunications for official technical review of the report and their suggested improvements, and to R.B. Stoner for suggestions on format and exposition.

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APPENDIX A. SUMMARIES OF MONTE CARLO SIMULATIONS

Each of Tables A.1-A.31 summarizes the results of simulating samples of dependent transmissions (trials) according to the Markov model of Section 1. All except two (A.25 and A.31) have 1000 samples each, and those two have 2000 each. Tables A.1-A.25 have confidence intervals calculated using $\hat{\lambda}$ (2.2) as the estimate of the conditional probability of an error given an error (λ), while Tables A.26-A.31 use the less biased estimator $\tilde{\lambda}$ (11.4)-(11.6).

The tables are further subdivided as follows:

Tables A.1-A.10: $\lambda=.3$, $\alpha=.05$ (90% confidence level)
Tables A.11-A.17: $\lambda=.3$, $\alpha=.025$ (95% confidence level)
Tables A.18-A.25: $\lambda=.8$, $\alpha=.05$
Tables A.26-A.27: $\lambda=.3$, $\alpha=.05$
Tables A.28-A.31: $\lambda=.8$, $\alpha=.05$

Within these groups the tables progress from large error rate p (P on the computer printouts forming these tables) and small sample size n (N on the printouts) to small p and large n.

These summaries are discussed in Section 11.

Table A-1. Summary of 1000 Monte Carlo Simulations

ALPHA=.050 N= 50 P=.5000 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	0
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	0
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	2
NO. OF RHO GREATER THAN 1	0
NO. OF S=0	0
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	0

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.50046E+00	.29247E+00	.29231E+00
MIN=	.32000E+00	0.	0.
MAX=	.66000E+00	.56113E+00	.55958E+00
SIGMA=	.46934E-01	.93775E-01	.94390E-01

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANO-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.93687E+00	.93766E+00	.91283E+00	.91283E+00	.91182E+00
1ST 500	.95000E+00	.95200E+00	.91800E+00	.91800E+00	.91600E+00
2ND 500	.92369E+00	.92369E+00	.90763E+00	.90763E+00	.90763E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.04390E+00	.10164E+01	.10210E+01	.10233E+01	.10234E+01
MIN=	.34641E+00	.93252E+00	.96226E+00	.96138E+00	.95902E+00
MAX=	.10107E+01	.11326E+01	.11352E+01	.11493E+01	.11613E+01
SIGMA=	.10569E+00	.29518E-01	.27134E-01	.28351E-01	.31403E-01

NO. OF ITERATIONS

MEAN=	.81323E+01	.82034E+01	.71954E+01
MIN=	.50000E+01	.40000E+01	.40000E+01
MAX=	.20000E+02	.20000E+02	.13000E+02
NO. OF FAILURES=	8	4	0

Table A-2. Summary of 1000 Monte Carlo Simulations

ALPHA=.050 N= 50 P=.3000 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	0
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	0
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PUO GREATER THAN 1	0
NO. OF S=0	0
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	0

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.30162E+00	.27845E+00	.27884E+00
MIN=	.10000E+00	-.18126E-14	0.
MAX=	.54000E+00	.59516E+00	.61224E+00
SIGMA=	.65523E-01	.11808E+00	.11911E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANDERSON-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.91700E+00	.90800E+00	.91800E+00	.91800E+00	.91500E+00
1ST 500	.91800E+00	.91000E+00	.91400E+00	.91400E+00	.90800E+00
2ND 500	.91600E+00	.90600E+00	.92200E+00	.92200E+00	.92200E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.97662E+00	.10172E+01	.97052E+00	.96750E+00	.96668E+00
MIN=	.66974E+00	.95145E+00	.90594E+00	.88923E+00	.90072E+00
MAX=	.15678E+01	.10614E+01	.10230E+01	.10244E+01	.10100E+01
SIGMA=	.13808E+00	.17154E-01	.19162E-01	.21591E-01	.17371E-01

NO. OF ITERATIONS

MEAN=	.54300E+01	.54130E+01	.54250E+01
MIN=	.30000E+01	.30000E+01	.30000E+01
MAX=	.10000E+02	.13000E+02	.12000E+02
NO. OF FAILURES=	0	0	0

Table A-3. Summary of 1000 Monte Carlo Simulations

ALPHA=.050 N= 100 P=.100 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	21
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	0
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PUI GREATER THAN 1	0
NO. OF S=0	1
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	0

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.10057E+00	.29766E+00	.26753E+00
MIN=	0.	0.	0.
MAX=	.25000E+00	.74667E+00	.75758E+00
SIGMA=	.37430E-01	.15151E+00	.15167E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANO-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.90691E+00	.91592E+00	.90891E+00	.91191E+00	.90691E+00
1ST 500	.91334E+00	.92386E+00	.91984E+00	.92585E+00	.91984E+00
2ND 500	.89400E+00	.90200E+00	.89800E+00	.89800E+00	.89400E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.12327E+01	.10273E+01	.96779E+00	.95750E+00	.95607E+00
MIN=	.88315E+00	.79199E+00	.84814E+00	.85815E+00	.87561E+00
MAX=	.25779E+01	.12533E+01	.10343E+01	.99559E+00	.99694E+00
SIGMA=	.20972E+00	.17364E-01	.16001E-01	.20547E-01	.20973E-01

NO. OF ITERATIONS

MEAN=	.33764E+01	.34454E+01	.33504E+01
MIN=	.20000E+01	.20000E+01	.20000E+01
MAX=	.60000E+01	.60000E+01	.60000E+01
NO. OF FAILURES=	0	0	0

Table A-4. Summary of 1000 Monte Carlo Simulations

ALPHA=.050 N= 100 P=.050 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	468
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	2
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PU GREATER THAN 1	0
NO. OF S=0	109
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	395

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.50163E-01	.20810E+00	.20836E+00
MIN=	0.	0.	0.
MAX=	.14000E+00	.77625E+00	.75756E+00
SIGMA=	.22941E-01	.22055E+00	.22064E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANDERSON-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.98641E+00	.94501E+00	.98204E+00	.97531E+00	.98092E+00
1ST 500	.98402E+00	.94749E+00	.98174E+00	.97489E+00	.98174E+00
2ND 500	.98875E+00	.94260E+00	.98234E+00	.97572E+00	.98013E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.12582E+01	.10903E+01	.98931E+00	.96223E+00	.95365E+00
MIN=	.93606E+00	.71003E+00	.89893E+00	.82976E+00	.84573E+00
MAX=	.27042E+01	.13657E+01	.10300E+01	.10227E+01	.99430E+00
SIGMA=	.33375E+00	.44464E-01	.24092E-01	.33557E-01	.36745E-01

NO. OF ITERATIONS

MEAN=	.32761E+01	.35163E+01	.31908E+01
MIN=	.30000E+01	.30000E+01	.30000E+01
MAX=	.60000E+01	.50000E+01	.50000E+01
NO. OF FAILURES=	0	0	0

Table A-5. Summary of 1000 Monte Carlo Simulations

ALPHA=.050 N= 500 P=.030 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	0
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	0
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF RHO GREATER THAN 1	0
NO. OF S=0	0
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	0

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.30032E-01	.27801E+00	.27787E+00
MIN=	.40000E-02	-.31643E-14	0.
MAX=	.66000E-01	.61974E+00	.62029E+00
SIGMA=	.10334E-01	.12216E+00	.12216E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANDERSON-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.92200E+00	.89500E+00	.91200E+00	.90800E+00	.90800E+00
1ST 500	.92200E+00	.89400E+00	.91200E+00	.90400E+00	.90400E+00
2ND 500	.92200E+00	.89600E+00	.91200E+00	.91200E+00	.91200E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.13168E+01	.10138E+01	.99144E+00	.98753E+00	.97862E+00
MIN=	.97160E+00	.10000E+01	.97948E+00	.97211E+00	.95288E+00
MAX=	.20064E+01	.10719E+01	.10212E+01	.10177E+01	.10070E+01
SIGMA=	.17414E+00	.85511E-02	.65244E-02	.68655E-02	.92532E-02

NO. OF ITERATIONS

MEAN=	.28590E+01	.29440E+01	.25260E+01
MIN=	.20000E+01	.20000E+01	.20000E+01
MAX=	.50000E+01	.40000E+01	.40000E+01
NO. OF FAILURES=	0	0	0

Table A-6. Summary of 1000 Monte Carlo Simulations

ALPHA=.050 N= 1000 P=.003 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	478
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	0
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PUI GREATER THAN 1	0
NO. OF S=0	116
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	414

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.29790E-02	.22091E+00	.22095E+00
MIN=	0.	0.	0.
MAX=	.15000E-01	.74989E+00	.75075E+00
SIGMA=	.23341E-02	.22929E+00	.22938E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANDERSON-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.96643E+00	.95646E+00	.96196E+00	.97964E+00	.98077E+00
1ST 500	.95636E+00	.96187E+00	.97987E+00	.97763E+00	.97987E+00
2ND 500	.96627E+00	.97325E+00	.96398E+00	.98169E+00	.98169E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.13167E+01	.13920E+01	.10383E+01	.10166E+01	.98356E+00
MIN=	.99197E+00	.10000E+01	.96130E+00	.98937E+00	.94296E+00
MAX=	.26390E+01	.12931E+01	.10973E+01	.11115E+01	.10093E+01
SIGMA=	.37172E+00	.92477E-01	.20643E-01	.23013E-01	.22524E-01

NO. OF ITERATIONS

MEAN=	.28360E+01	.27511E+01	.28326E+01
MIN=	.20000E+01	.20000E+01	.20000E+01
MAX=	.40000E+01	.40000E+01	.40000E+01
NO. OF FAILURES=	0	0	0

Table A-7. Summary of 1000 Monte Carlo Simulations

ALPHA=.250 M= 1000 P=.001 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	414
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	0
NO. OF ABS. VALUE OF PHO HAT GREATER THAN 1	0
NO. OF PHO GREATER THAN 1	0
NO. OF S=0	519
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	400

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.99200E-03	.19994E+00	.20021E+00
MIN=	0.	0.	0.
MAX=	.80000E-02	.83326E+00	.83417E+00
SIGMA=	.14064E-02	.25362E+00	.25394E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANDERSON-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.98960E+00	.91268E+00	.98960E+00	.87734E+00	.95218E+00
1ST 500	.99149E+00	.88511E+00	.99149E+00	.83830E+00	.94468E+00
2ND 500	.98780E+00	.93992E+00	.98780E+00	.91463E+00	.95935E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.13114E+01	.11269E+01	.10460E+01	.10323E+01	.98668E+00
MIN=	.99700E+00	.10003E+01	.99873E+00	.99398E+00	.94296E+00
MAX=	.33049E+01	.13306E+01	.11108E+01	.11258E+01	.10093E+01
SIGMA=	.44033E+00	.56629E-01	.14876E-01	.25901E-01	.21899E-01

NO. OF ITERATIONS

MEAN=	.29647E+01	.29127E+01	.30042E+01
MIN=	.20000E+01	.20000E+01	.20000E+01
MAX=	.30000E+01	.40000E+01	.50000E+01
NO. OF FAILURES=	0	0	0

Table A-8. Summary of 1000 Monte Carlo Simulations

ALPHA=.050 N= 1000 P=.0003 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	163
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	0
NO. OF RES. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PUI GREATER THAN 1	0
NO. OF C=0	826

NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE 163

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.20000E-03	.19065E-01	.19090E+00
MIN=	0.	0.	0.
MAX=	.60000E-02	.83826E+00	.83417E+00
SIGMA=	.72319E-03	.26418E+00	.26451E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANDERSON-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.95678E+00	.81468E+00	.99425E+00	.58621E+00	.87356E+00
1ST 500	.91867E+00	.76842E+00	.98850E+00	.55208E+00	.83333E+00
2ND 500	.96154E+00	.85897E+00	.10000E+01	.62821E+00	.92308E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.13068E+01	.11432E+01	.13468E+01	.10414E+01	.98964E+00
MIN=	.99800E+00	.10000E+01	.20000E+01	.10000E+01	.94296E+00
MAX=	.33249E+01	.13306E+01	.11100E+01	.11258E+01	.10090E+01
SIGMA=	.45760E+00	.56726E-01	.13466E-01	.24178E-01	.19560E-01

NO. OF ITERATIONS

MEAN=	.30000E+01	.30517E+01	.30172E+01
MIN=	.30000E+01	.20000E+01	.30000E+01
MAX=	.30000E+01	.40000E+01	.40000E+01
NO. OF FAILURES=	0	0	0

Table A-9. Summary of 1000 Monte Carlo Simulations

ALPHA=.050 N= 1000 P=.0003 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	160
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	0
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PUB GREATER THAN 1	0
NO. OF S=0	833
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	160

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.23300E-03	.13220E+00	.13237E+00
MIN=	0.	0.	0.
MAX=	.40000E-02	.74989E+00	.75075E+00
SIGMA=	.59250E-03	.23745E+00	.23775E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANO-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.95808E+00	.86226E+00	.10000E+01	.71257E+00	.92814E+00
1ST 500	.95652E+00	.83696E+00	.10000E+01	.68478E+00	.92391E+00
2ND 500	.96000E+00	.89333E+00	.10000E+01	.74667E+00	.93333E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.12111E+01	.11350E+01	.10449E+01	.10389E+01	.99403E+00
MIN=	.99800E+00	.10000E+01	.10000E+01	.10000E+01	.94550E+00
MAX=	.26390E+01	.12931E+01	.16973E+01	.11115E+01	.10090E+01
SIGMA=	.39777E+00	.49932E-01	.11669E-01	.21022E-01	.16795E-01

NO. OF ITERATIONS

MEAN=	.30000E+01	.30479E+01	.30060E+01
MIN=	.30000E+01	.20000E+01	.30000E+01
MAX=	.30000E+01	.40000E+01	.40000E+01
NO. OF FAILURES=	0	0	0

Table A-10. Summary of 1000 Monte Carlo Simulations

ALPHA=.350 N= 1000 P=.0003 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	181
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	0
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PUO GREATER THAN 1	0
NO. OF S=0	816
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	100

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.28900E-03	.15313E+00	.15332E+00
MIN=	0.	0.	0.
MAX=	.60000E-02	.74991E+00	.80060E+00
SIGMA=	.74983E-03	.25292E+00	.25324E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANDERSON-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.97826E+00	.65326E+00	.10000E+01	.70109E+00	.88043E+00
1ST 500	.97936E+00	.65567E+00	.10000E+01	.70103E+00	.87629E+00
2ND 500	.97731E+00	.65057E+00	.10000E+01	.70115E+00	.88506E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.12545E+01	.11365E+01	.11460E+01	.10382E+01	.99205E+00
MIN=	.99800E+00	.10000E+01	.10000E+01	.99684E+00	.94296E+00
MAX=	.29909E+01	.13140E+01	.11960E+01	.11196E+01	.10090E+01
SIGMA=	.45943E+00	.52177E-01	.13783E-01	.23103E-01	.18653E-01

NO. OF ITERATIONS

MEAN=	.29946E+01	.29946E+01	.30435E+01
MIN=	.20000E+01	.20000E+01	.20000E+01
MAX=	.30000E+01	.40000E+01	.50000E+01
NO. OF FAILURES=	0	0	0

Table A-11. Summary of 1000 Monte Carlo Simulations

ALPHA=.025 N= 50 P=.5000 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	0
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	0
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	18
NO. OF PUO GREATER THAN 1	0
NO. OF S=0	0
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	0

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.50048E+00	.29247E+00	.29231E+00
MIN=	.32000E+00	0.	0.
MAX=	.66000E+00	.56113E+00	.55958E+00
SIGMA=	.46934E-01	.93775E-01	.94390E-01

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANO-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.97556E+00	.96640E+00	.94399E+00	.94399E+00	.94908E+00
1ST 500	.97963E+00	.97556E+00	.95519E+00	.95519E+00	.95927E+00
2ND 500	.97149E+00	.95723E+00	.93279E+00	.93279E+00	.93890E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.64698E+00	.10133E+01	.10197E+01	.10190E+01	.10252E+01
MIN=	.34641E+00	.93324E+00	.96435E+00	.95900E+00	.95783E+00
MAX=	.10107E+01	.11153E+01	.10950E+01	.10966E+01	.11341E+01
SIGMA=	.10328E+00	.26192E-01	.22313E-01	.24262E-01	.30523E-01

NO. OF ITERATIONS

MEAN=	.12221E+02	.12312E+02	.97770E+01
MIN=	.60000E+01	.50000E+01	.40000E+01
MAX=	.20000E+02	.20000E+02	.19000E+02
NO. OF FAILURES=	60	77	0

Table A-12. Summary of 1000 Monte Carlo Simulations

ALPHA=.025 N= 50 P=.3000 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	3
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	3
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PUO GREATER THAN 1	0
NO. OF S=0	0
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	0

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.30162E+00	.27845E+00	.27804E+00
MIN=	.10000E+00	-.18126E-14	0.
MAX=	.54000E+00	.59516E+00	.61224E+00
SIGMA=	.65523E-01	.11808E+00	.11911E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	AND-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.96100E+00	.95900E+00	.96300E+00	.96300E+00	.96300E+00
1ST 500	.96200E+00	.96600E+00	.96000E+00	.96000E+00	.96000E+00
2ND 500	.96000E+00	.95200E+00	.96600E+00	.96600E+00	.96600E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.97662E+00	.10259E+01	.96507E+00	.96039E+00	.96043E+00
MIN=	.66974E+00	.95595E+00	.88596E+00	.88866E+00	.87326E+00
MAX=	.15678E+01	.10808E+01	.10685E+01	.10710E+01	.10069E+01
SIGMA=	.13808E+00	.16906E-01	.20109E-01	.20637E-01	.18043E-01

NO. OF ITERATIONS

MEAN=	.67940E+01	.67910E+01	.69310E+01
MIN=	.40000E+01	.40000E+01	.40000E+01
MAX=	.20000E+02	.18000E+02	.14000E+02
NO. OF FAILURES=	1	0	0

Table A-13. Summary of 1000 Monte Carlo Simulations

ALPHA=.025 N= 100 P=.100 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	295
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	158
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF RHO GREATER THAN 1	0
NO. OF S=0	1
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	10

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.10066E+00	.26796E+00	.25794E+00
MIN=	0.	0.	0.
MAX=	.25903E+00	.74887E+00	.75758E+00
SIGMA=	.37427E-01	.15083E+00	.15111E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANDERSON-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.94595E+00	.95095E+00	.94895E+00	.94795E+00	.94795E+00
1ST 500	.95792E+00	.96794E+00	.96192E+00	.95992E+00	.95992E+00
2ND 500	.93410E+00	.93400E+00	.93600E+00	.93600E+00	.93600E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.12737E+01	.10491E+01	.99197E+00	.97480E+00	.95166E+00
MIN=	.86318E+00	.10000E+01	.90317E+00	.89527E+00	.85420E+00
MAX=	.25779E+01	.13916E+01	.11262E+01	.10959E+01	.99256E+00
SIGMA=	.20356E+00	.24907E-01	.57580E-01	.35959E-01	.28939E-01

NO. OF ITERATIONS

MEAN=	.40531E+01	.40180E+01	.37608E+01
MIN=	.30000E+01	.30000E+01	.30000E+01
MAX=	.20000E+02	.15000E+02	.80000E+01
NO. OF FAILURES=	19	0	0

Table A-14. Summary of 1000 Monte Carlo Simulations

ALPHA=.025 N= 500 P=.100 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	0
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	0
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PUT GREATER THAN 1	0
NO. OF S=0	0
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	0

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.10003E+00	.29558E+00	.29559E+00
MIN=	.52000E-01	.10237E+00	.10277E+00
MAX=	.15200E+00	.49961E+00	.50100E+00
SIGMA=	.16927E-01	.66523E-01	.66471E-01

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANDERSON-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.94900E+00	.95500E+00	.95300E+00	.95300E+00	.95300E+00
1ST 500	.94200E+00	.95000E+00	.94200E+00	.94200E+00	.94200E+00
2ND 500	.95600E+00	.96000E+00	.96400E+00	.96400E+00	.96400E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.12522E+01	.10008E+01	.98283E+00	.98252E+00	.98076E+00
MIN=	.10000E+01	.97527E+00	.95527E+00	.95477E+00	.95298E+00
MAX=	.16066E+01	.10272E+01	.10132E+01	.10128E+01	.10110E+01
SIGMA=	.99626E-01	.98847E-02	.97603E-02	.97772E-02	.97959E-02

NO. OF ITERATIONS

MEAN=	.22340E+01	.22490E+01	.23240E+01
MIN=	.20000E+01	.20000E+01	.20000E+01
MAX=	.90000E+01	.50000E+01	.50000E+01
NO. OF FAILURES=	0	0	0

Table A-15. Summary of 1000 Monte Carlo Simulations

ALPHA=.025 N= 100 P=.030 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	860
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	514
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PU0 GREATER THAN 1	0
NO. OF S=0	100
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	406

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.30140E-01	.21249E+00	.21295E+00
MIN=	0.	0.	0.
MAX=	.14000E+00	.74887E+00	.75758E+00
SIGMA=	.22187E-01	.22773E+00	.22822E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	AND-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.99778E+00	.97667E+00	.99889E+00	.99778E+00	.99444E+00
1ST 500	.99779E+00	.98009E+00	.10000E+01	.99779E+00	.99558E+00
2ND 500	.99777E+00	.97321E+00	.99777E+00	.99777E+00	.99330E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.12649E+01	.11444E+01	.10019E+01	.99818E+00	.94909E+00
MIN=	.93968E+00	.10000E+01	.87959E+00	.89335E+00	.83512E+00
MAX=	.25779E+01	.15720E+01	.11161E+01	.10899E+01	.98749E+00
SIGMA=	.34723E+00	.69928E-01	.41150E-01	.32732E-01	.38356E-01

NO. OF ITERATIONS

MEAN=	.39211E+01	.36789E+01	.35478E+01
MIN=	.30000E+01	.30000E+01	.30000E+01
MAX=	.20000E+02	.15000E+02	.50000E+01
NO. OF FAILURES=	21	0	0

Table A-16. Summary of 1000 Monte Carlo Simulations

ALPHA=.025 N= 50] P=.030 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	110
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	25
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PUO GREATER THAN 1	0
NO. OF S=0	0
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	0

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.30038E-01	.27796E+00	.27783E+00
MIN=	.40000E-02	-.31643E-14	0.
MAX=	.66000E-01	.61874E+00	.62029E+00
SIGMA=	.10338E-01	.12201E+00	.12199E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANDERSON-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.95300E+00	.94800E+00	.93100E+00	.95100E+00	.95100E+00
1ST 500	.94800E+00	.94200E+00	.94800E+00	.94800E+00	.94800E+00
2ND 500	.95800E+00	.95400E+00	.95400E+00	.95400E+00	.95400E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.13167E+01	.10308E+01	.10045E+01	.99563E+00	.97909E+00
MIN=	.97160E+00	.10000E+01	.96543E+00	.97933E+00	.95763E+00
MAX=	.20064E+01	.11091E+01	.11942E+01	.11299E+01	.10073E+01
SIGMA=	.17389E+00	.13090E-01	.46693E-01	.18466E-01	.84578E-02

NO. OF ITERATIONS

MEAN=	.60930E+01	.27370E+01	.27920E+01
MIN=	.20000E+01	.20000E+01	.20000E+01
MAX=	.20000E+02	.20000E+02	.50000E+01
NO. OF FAILURES=	20	1	0

Table A-17. Summary of 1000 Monte Carlo Simulations

ALPHA=.025 N= 1000 P=.003 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	852
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	487
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PUO GREATER THAN 1	0
NO. OF S=0	121
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	434

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.30120E-02	.22220E+00	.22212E+00
MIN=	0.	0.	0.
MAX=	.15000E-01	.79991E+00	.80080E+00
SIGMA=	.23705E-02	.22799E+00	.22799E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	AND-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.99431E+00	.97725E+00	.99772E+00	99204E+00	.98862E+00
1ST 500	.99318E+00	.97273E+00	.99773E+00	.99318E+00	.98636E+00
2ND 500	.99544E+00	.98178E+00	.99772E+00	.99089E+00	.99089E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.13177E+01	.11437E+01	.10575E+01	.10638E+01	.98732E+00
MIN=	.99197E+00	.10000E+01	.98492E+00	.10000E+01	.95232E+00
MAX=	.29909E+01	.14659E+01	.11708E+01	.11661E+01	.10098E+01
SIGMA=	.37071E+00	.77913E-01	.25516E-01	.25978E-01	.20965E-01

NO. OF ITERATIONS

MEAN=	.29852E+01	.25119E+01	.29761E+01
MIN=	.20000E+01	.20000E+01	.20000E+01
MAX=	.40000E+01	.40000E+01	.40000E+01
NO. OF FAILURES=	0	0	0

Table A-18. Summary of 1000 Monte Carlo Simulations

ALPHA=.050 N= 50 P=.5000 LAMBDA=.800

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	27
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	0
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	105
NO. OF PUO GREATER THAN 1	38
NO. OF S=0	0
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	2

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.50462E+00	.77969E+00	.77936E+00
MIN=	.16000E+00	.32878E+00	.34014E+00
MAX=	.92000E+00	.96449E+00	.97604E+00
SIGMA=	.14274E+00	.93772E-01	.97330E-01

FRACTION OF CONFIDENCE INTERVALS COVERING P

	AND-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.90838E+00	.89944E+00	.87374E+00	.87374E+00	.87933E+00
1ST 500	.91209E+00	.90549E+00	.88352E+00	.88352E+00	.89011E+00
2ND 500	.90455E+00	.89318E+00	.86364E+00	.86364E+00	.86818E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.19245E+01	.92828E+00	.89377E+00	.87600E+00	.89743E+00
MIN=	.10000E+01	.59224E+00	.59224E+00	.59224E+00	.56026E+00
MAX=	.38296E+01	.11532E+01	.10436E+01	.10318E+01	.12105E+01
SIGMA=	.37739E+00	.56187E-01	.41884E-01	.51506E-01	.78863E-01

NO. OF ITERATIONS

MEAN=	.15103E+02	.14993E+02	.16135E+02
MIN=	40000E+01	.40000E+01	.20000E+01
MAX=	.20000E+02	.20000E+02	.20000E+02
NO. OF FAILURES=	307	298	399

Table A-19. Summary of 1000 Monte Carlo Simulations

ALPHA=.050 N= 50 P=.3000 LAMBDA=.800

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	388
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	3
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	54
NO. OF PUO GREATER THAN 1	56
NO. OF S=0	5
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	201

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.30376E+00	.75013E+00	.74943E+00
MIN=	0.	0.	0.
MAX=	.86000E+00	.97781E+00	.97604E+00
SIGMA=	.15357E+00	.15073E+00	.15520E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	AND-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.87991E+00	.86079E+00	.85760E+00	.85760E+00	.84060E+00
1ST 500	.86441E+00	.85159E+00	.85381E+00	.85381E+00	.83686E+00
2ND 500	.89552E+00	.86994E+00	.86141E+00	.86141E+00	.84435E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.23210E+01	.10066E+01	.86620E+00	.80710E+00	.84846E+00
MIN=	.95917E+00	.66346E+00	.66346E+00	.58745E+00	.31925E+00
MAX=	.69051E+01	.13842E+01	.11630E+01	.11630E+01	.12105E+01
SIGMA=	.64323E+00	.83381E-01	.53948E-01	.70952E-01	.13516E+00

NO. OF ITERATIONS

MEAN=	.12385E+02	.12375E+02	.12371E+02
MIN=	.30000E+01	.40000E+01	.20000E+01
MAX=	.20000E+02	.20000E+02	.20000E+02
NO. OF FAILURES=	172	185	332

Table A-20. Summary of 1000 Monte Carlo Simulations

ALPHA=.050 N= 100 P=.1000 LAMBDA=.800

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	742
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	15
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PUC GREATER THAN 1	3
NO. OF S=0	87
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	557

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.99551E-01	.68364E+00	.68164E+00
MIN=	0.	0.	0.
MAX=	.55000E+00	.97174E+00	.97403E+00
SIGMA=	.84505E-01	.24048E+00	.24189E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANDERSON-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.84337E+00	.84447E+00	.83461E+00	.81818E+00	.82585E+00
1ST 500	.83772E+00	.83553E+00	.82675E+00	.81360E+00	.81579E+00
2ND 500	.84902E+00	.85339E+00	.84245E+00	.82276E+00	.83589E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.25230E+01	.10971E+01	.90975E+00	.84168E+00	.83941E+00
MIN=	.96954E+00	.89763E+00	.69401E+00	.59428E+00	.22867E+00
MAX=	.62485E+01	.14711E+01	.10368E+01	.10227E+01	.29386E+01
SIGMA=	.90794E+00	.14344E+00	.11298E+00	.11865E+00	.14756E+00

NO. OF ITERATIONS

MEAN=	.59198E+01	.60846E+01	.54308E+01
MIN=	.30000E+01	.30000E+01	.30000E+01
MAX=	.20000E+02	.20000E+02	.20000E+02
NO. OF FAILURES=	10	9	12

Table A-21. Summary of 1000 Monte Carlo Simulations

ALPHA=.050 N= 100 P=.030 LAMBDA=.800

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0, 0.5)	461
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	24
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	1
NO. OF RHO GREATER THAN 1	3
NO. OF S=0	525
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	438

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.30730E-01	.61815E+00	.61614E+00
MIN=	0.	0.	0.
MAX=	.38002E+00	.97005E+00	.97527E+00
SIGMA=	.51645E-01	.31142E+00	.31105E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANDERSON-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.99789E+00	.86498E+00	.99367E+00	.81646E+00	.92616E+00
1ST 500	.10000E+01	.89492E+00	.99613E+00	.82937E+00	.92460E+00
2ND 500	.99560E+00	.84234E+00	.99099E+00	.80180E+00	.92793E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.25790E+01	.11821E+01	.91721E+00	.86913E+00	.84331E+00
MIN=	.97382E+00	.92685E+00	.69025E+00	.58900E+00	.39780E+00
MAX=	.23778E+01	.15711E+01	.10173E+01	.10227E+01	.29386E+01
SIGMA=	.11826E+01	.92160E-01	.76489E-01	.10344E+00	.16377E+00

	NO. OF ITERATIONS		
MEAN=	.57707E+01	.59002E+01	.52463E+01
MIN=	.30000E+01	.30000E+01	.30000E+01
MAX=	.20000E+02	.20000E+02	.20000E+02
NO. OF FAILURES=	9	6	21

Table A-22. Summary of 1000 Monte Carlo Simulations

ALPHA=.750 N= 500 P=.37 LAMBDA=.800

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	732
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	3
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF RHO GREATER THAN 1	0
NO. OF S=0	41
NO. OF S=1	0
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	548

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.36222E-01	.72802E+00	.72753E+00
MIN=	0.	0.	0.
MAX=	.17231E+00	.97013E+00	.95646E+00
SIGMA=	.21411E-01	.19092E+00	.19088E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	AND-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.89990E+00	.89051E+00	.89468E+00	.88947E+00	.88947E+00
1ST 500	.91952E+00	.89189E+00	.90644E+00	.90229E+00	.90229E+00
2ND 500	.88121E+00	.88912E+00	.88285E+00	.87657E+00	.87657E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.28419E+01	.11182E+01	.10734E+01	.97794E+00	.94216E+00
MIN=	.99708E+00	.10000E+01	.88191E+00	.81590E+00	.84516E+00
MAX=	.79796E+01	.16341E+01	.11721E+01	.12521E+01	.10345E+01
SIGMA=	.88472E+00	.81970E-01	.36940E-01	.39968E-01	.18985E-01

NO. OF ITERATIONS

MEAN=	.35610E+01	.34463E+01	.31283E+01
MIN=	.20000E+01	.20000E+01	.20000E+01
MAX=	.20000E+02	.60000E+01	.60000E+01
NO. OF FAILURES=	11	0	0

Table A-23. Summary of 1000 Monte Carlo Simulations

ALPHA=.050 N= 1000 P=.003 LAMBDA=.600

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	413
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	4
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PU[illegible] GREATER THAN 1	0
NO. OF S=0	582
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	407

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.29931E-02	.64772E+00	.64718E+00
MIN=	0.	0.	0.
MAX=	.46[illegible]E-01	.95831E+00	.95929E+00
SIGMA=	.54791E-02	.29038E+00	.29029E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANDERSON-BURSTEIN	NORMAL	EDGEWORTH3	EDGEWORTH4	PEARSON
ALL 1000	.10000E+01	.96603E+00	.99761E+00	.77512E+00	.89952E+00
1ST 500	.10000E+01	.96693E+00	.10000E+01	.78792E+00	.85644E+00
2ND 500	.10000E+01	.92130E+00	.99537E+00	.83796E+00	.93981E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.27824E+01	.12323E+01	.10768E+01	.10671E+01	.97623E+00
MIN=	.39900E+00	.10000E+01	.10000E+01	.93854E+00	.92202E+00
MAX=	.67694E+01	.16061E+01	.12463E+01	.13396E+01	.10780E+01
SIGMA=	.12552E+01	.10144E+00	.30247E-01	.54320E-01	.22454E-01

NO. OF ITERATIONS

MEAN=	.31770E+01	.33636E+01	.34091E+01
MIN=	.20000E+01	.20000E+01	.20000E+01
MAX=	.60000E+01	.60000E+01	.50000E+01
NO. OF FAILURES=	0	0	0

Table A-24. Summary of 1000 Monte Carlo Simulations

ALPHA=.050 N= 100 P=.001 LAMBDA=.800

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	169
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	3
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PUC GREATER THAN 1	0
NO. OF S=1	828
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	169

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.39700E-03	.60249E+00	.60074E+00
MIN=	0.	0.	0.
MAX=	.29100E-01	.96550E+00	.96468E+00
SIGMA=	.92142E-02	.32338E+00	.32286E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANO-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.10000E+01	.71930E+00	.10000E+01	.59884E+00	.73837E+00
1ST 500	.10000E+01	.69920E+00	.10000E+01	.60215E+00	.74194E+00
2ND 500	.10000E+01	.72152E+00	.10000E+01	.59494E+00	.73418E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.87113E+01	.12513E+01	.10779E+01	.10803E+01	.98302E+00
MIN=	.99000E+00	.10000E+01	.10000E+01	.99000E+00	.94322E+00
MAX=	.74357E+01	.16361E+01	.12463E+01	.13396E+01	.10780E+01
SIGMA=	.14313E+01	.10219E+00	.33130E-01	.50424E-01	.20422E-01

NO. OF ITERATIONS

MEAN=	.32267E+01	.34012E+01	.34244E+01
MIN=	.30000E+01	.30000E+01	.20000E+01
MAX=	.60000E+01	.40000E+01	.50000E+01
NO. OF FAILURES=	0	0	0

Table A-25. Summary of 2000 Monte Carlo Simulations

ALPHA=.05L N= 1000 P=.0003 LAMBDA=.900

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	111
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	0
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PUL GREATER THAN 1	0
NO. OF S=0	1889
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	111

	P HAT	LAMBDA HAT	LAMBDA STAR
MEAN=	.2900[illegible]E-03	.55873E+00	.55937E+00
MIN=	..	..	0.
MAX=	.33000E-01	.96968E+00	.97067E+00
SIGMA=	.18309E-02	.34467E+00	.34504E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANO-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 2000	.10[illegible]E+01	.40641E+00	.10000E+01	.24324E+00	.51351E+00
1ST 1000	.10000E+01	.39216E+00	.10000E+01	.25490E+00	.52941E+00
2ND 1000	.10000E+01	.41667E+00	.10000E+01	.23333E+00	.50000E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.26101E+01	.12486E+01	.10747E+01	.10787E+01	.98486E+00
MIN=	.99908E+00	.10000E+01	.10000E+01	.99934E+00	.94322E+00
MAX=	.79240E+01	.16931E+01	.11150E+01	.11276E+01	.10036E+01
SIGMA=	.15541E+01	.96508E-01	.28431E-01	.38032E-01	.15456E-01

NO. OF ITERATIONS -

MEAN=	.31622E+01	.33784E+01	.34505E+01
MIN=	.30000E+01	.30000E+01	.30000E+01
MAX=	.60000E+01	.60000E+01	.50000E+01
NO. OF FAILURES=	0	0	0

Table A-26. Summary of 1000 Monte Carlo Simulations with Intervals Calculated From Lambda Tilde ($\tilde{\lambda}$)

ALPHA=.150 N= 100 P=.1000 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	38
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	1
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PUO GREATER THAN 1	0
NO. OF S=0	1
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	21

	P HAT	LAMBDA HAT	LAMBDA STAR	LAMBDA TILDE
MEAN=	.10057E+00	.26766E+00	.26753E+00	.30146E+00
MIN=	0.	0.	0.	0.
MAX=	.25003E+00	.74887E+00	.75758E+00	.85731E+00
SIGMA=	.37483E-01	.15151E+00	.15167E+00	.16926E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANDERSON-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.91291E+00	.91992E+00	.91592E+00	.91592E+00	.91391E+00
1ST 500	.92585E+00	.93788E+00	.92986E+00	.92986E+00	.92585E+00
2ND 500	.90000E+00	.90200E+00	.90200E+00	.90200E+00	.90200E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.12341E+01	.10289E+01	.96438E+00	.95204E+00	.95263E+00
MIN=	.98316E+00	.99153E+00	.90701E+00	.80356E+00	.81913E+00
MAX=	.35297E+01	.14469E+01	.10323E+01	.99553E+00	.99694E+00
SIGMA=	.36988E+00	.23200E-01	.20333E-01	.24946E-01	.25101E-01

NO. OF ITERATIONS

MEAN=	.34324E+01	.33524E+01	.35125E+01
MIN=	.20000E+01	.20000E+01	.20000E+01
MAX=	.70000E+01	.70000E+01	.11000E+02
NO. OF FAILURES=	0	0	0

Table A-27. Summary of 1000 Monte Carlo Simulations with Intervals Calculated From Lambda Tilde ($\tilde{\lambda}$)

ALPHA=.05 N= 1000 P=.0003 LAMBDA=.300

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	104
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,1.5)	53
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF RUC GREATER THAN 1	0
NO. OF S=0	826
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	163

	P HAT	LAMBDA HAT	LAMBDA STAR	LAMBDA TILDE
MEAN=	.26200E-03	.19065E+00	.19090E+00	.24066E+00
MIN=	0.	0.	0.	0.
MAX=	.60000E-02	.83326E+00	.83417E+00	.90075E+00
SIGMA=	.72319E-03	.26418E+00	.26451E+00	.33053E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	AND-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.93678E+00	.88506E+00	.10000E+01	.89080E+00	.88506E+00
1ST 500	.91667E+00	.85417E+00	.10000E+01	.86450E+00	.85417E+00
2ND 500	.96194E+00	.92308E+00	.10000E+01	.92308E+00	.92308E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.14953E+01	.11951E+01	.10716E+01	.10670E+01	.10019E+01
MIN=	.99600E+00	.10000E+01	.10000E+01	.10000E+01	.94778E+00
MAX=	.43623E+01	.15593E+01	.12213E+01	.13113E+01	.10622E+01
SIGMA=	.74642E+00	.13584E+00	.53407E-01	.92685E-01	.19087E-01

NO. OF ITERATIONS

MEAN=	.29085E+01	.32356E+01	.30805E+01
MIN=	.20000E+01	.30000E+01	.20000E+01
MAX=	.40000E+01	.40000E+01	.40000E+01
NO. OF FAILURES=	0	0	0

Table A-28. Summary of 1000 Monte Carlo Simulations with Intervals Calculated From Lambda Tilde ($\tilde{\lambda}$)

ALPHA=.050 N= 50 P=.5000 LAMBDA=.800

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	479
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	41
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	72
NO. OF PUC GREATER THAN 1	110
NO. OF S=0	6
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	270

	P HAT	LAMBDA HAT	LAMBDA STAR	LAMBDA TILDE
MEAN=	.50376E+00	.75013E+00	.74943E+00	.80469E+00
MIN=	0.	0.	0.	0.
MAX=	.86000E+00	.97781E+00	.97604E+00	.98057E+00
SIGMA=	.15357E+00	.15073E+00	.15520E+00	.13576E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANO-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.91983E+00	.89491E+00	.90358E+00	.90358E+00	.88732E+00
1ST 500	.90909E+00	.88312E+00	.89177E+00	.89177E+00	.87229E+00
2ND 500	.93059E+00	.90672E+00	.91540E+00	.91540E+00	.90239E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.27165E+01	.10210E+01	.84941E+00	.78361E+00	.81075E+00
MIN=	.95917E+00	.65685E+00	.65885E+00	.57842E+00	.45878E+00
MAX=	.73968E+01	.16079E+01	.11792E+01	.11792E+01	.12524E+01
SIGMA=	.79505E+00	.11318E+00	.72730E-01	.96362E-01	.16804E+00

NO. OF ITERATIONS

MEAN=	.14386E+02	.14534E+02	.13100E+02
MIN=	.30000E+01	.40000E+01	.20000E+01
MAX=	.20000E+02	.20000E+02	.20000E+02
NO. OF FAILURES=	240	272	345

Table A-29. Summary of 1000 Monte Carlo Simulations with Intervals Calculated From Lambda Tilde ($\tilde{\lambda}$)

ALPHA=.050 N= 100 P=.1000 LAMBDA=.800

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	808
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	160
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	2
NO. OF PUC GREATER THAN 1	7
NO. OF S=0	87
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	703

	P HAT	LAMBDA HAT	LAMBDA STAR	LAMBDA TILDE
MEAN=	.99550E-01	.68384E+00	.68164E+00	.75175E+00
MIN=	0.	0.	0.	0.
MAX=	.55000E+00	.97174E+00	.97403E+00	.97687E+00
SIGMA=	.84505E-01	.24048E+00	.24189E+00	.24257E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANDERSON-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.90889E+00	.85950E+00	.90340E+00	.88804E+00	.88913E+00
1ST 500	.90110E+00	.85714E+00	.89011E+00	.87912E+00	.87473E+00
2ND 500	.91667E+00	.86184E+00	.91667E+00	.89693E+00	.90351E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.30934E+01	.11767E+01	.88876E+00	.82638E+00	.83051E+00
MIN=	.96954E+00	.89727E+00	.68540E+00	.59147E+00	.27135E+00
MAX=	.83616E+01	.17374E+01	.13618E+01	.13618E+01	.34758E+01
SIGMA=	.10349E+01	.14139E+00	.75625E-01	.13221E+00	.17943E+00

NO. OF ITERATIONS

MEAN=	.74148E+01	.76881E+01	.69347E+01
MIN=	.30000E+01	.30000E+01	.30000E+01
MAX=	.20000E+02	.20000E+02	.20000E+02
NO. OF FAILURES=	21	28	51

Table A-30. Summary of 1000 Monte Carlo Simulations with Intervals Calculated From Lambda Tilde ($\tilde{\lambda}$)

ALPHA=.050 N= 1000 P=.0030 LAMBDA=.800

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	413
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	234
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF PUO GREATER THAN 1	0
NO. OF S=0	582
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	413

	P HAT	LAMBDA HAT	LAMBDA STAR	LAMBDA TILDE
MEAN=	.23930E-02	.64772E+00	.64718E+00	.72130E+00
MIN=	0.	0.	0.	0.
MAX=	.46000E-01	.99831E+00	.95929E+00	.96315E+00
SIGMA=	.54791E-02	.29038E+00	.29029E+00	.30233E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANDERSON-BURSTEIN	NORMAL	EDGEWORTH2	EDGEWORTH4	PEARSON
ALL 1000	.10000E+01	.89474E+00	.99761E+00	.81340E+00	.92346E+00
1ST 500	.10000E+01	.84053E+00	.10000E+01	.74752E+00	.88138E+00
2ND 500	.10000E+01	.93981E+00	.99537E+00	.87500E+00	.95374E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.34460E+01	.13696E+01	.11330E+01	.11694E+01	.10124E+01
MIN=	.39800E+00	.10000E+01	.97513E+00	.93791E+00	.92252E+00
MAX=	.72094E+01	.19609E+01	.14295E+01	.15525E+01	.12096E+01
SIGMA=	.14824E+01	.18173E+00	.76347E-01	.12589E+00	.44351E-01

NO. OF ITERATIONS

MEAN=	.35000E+01	.37297E+01	.35450E+01
MIN=	.20000E+01	.30000E+01	.20000E+01
MAX=	.60000E+01	.70000E+01	.50000E+01
NO. OF FAILURES=	0	0	0

Table A-31. Summary of 2000 Monte Carlo Simulations with Intervals Calculated From Lambda Tilde ($\tilde{\lambda}$)

ALPHA=.050 N= 1000 P=.0003 LAMBDA=.800

NO. OF SAMPLES WITH ALPHA LI OUTSIDE OF (0,0.5)	111
NO. OF SAMPLES WITH ALPHA LI PRIME OUTSIDE OF (0,0.5)	75
NO. OF ABS. VALUE OF RHO HAT GREATER THAN 1	0
NO. OF FUO GREATER THAN 1	0
NO. OF S=0	1889
NO. OF SAMPLES WITH BETA LI BEYOND PEARSON-HARTLEY TABLE	111

	P HAT	LAMBDA HAT	LAMBDA STAR	LAMBDA TILDE
MEAN=	.29000E-03	.55873E+00	.55937E+00	.63114E+00
MIN=	0.	0.	0.	0.
MAX=	.33000E-01	.99968E+00	.97067E+00	.97176E+00
SIGMA=	.18305E-02	.34467E+00	.34504E+00	.37138E+00

FRACTION OF CONFIDENCE INTERVALS COVERING P

	ANO-BURSTEIN	NORMAL	EDGEWORTH3	EDGEWORTH4	PEARSON
ALL 2000	.10000E+01	.66067E+00	.10000E+01	.91892E+00	.66667E+00
1ST 1000	.10000E+01	.66667E+00	.10000E+01	.88235E+00	.66667E+00
2ND 1000	.10000E+01	.66667E+00	.10000E+01	.95000E+00	.66667E+00

RATIO OF CONFIDENCE INTERVAL LENGTH TO ANDERSON-BURSTEIN LENGTH
(FIRST COLUMN IS RATIO OF ANDERSON-BURSTEIN LENGTH
TO ANDERSON-BURSTEIN LENGTH WITH INDEPENDENCE ASSUMED)

MEAN=	.32312E+01	.13309E+01	.11402E+01	.11924E+01	.10243E+01
MIN=	.99900E+00	.10000E+01	.10000E+01	.10000E+01	.95522E+00
MAX=	.82145E+01	.15621E+01	.12208E+01	.13113E+01	.10625E+01
SIGMA=	.18111E+01	.17764E+00	.73930E-01	.11536E+00	.30232E-01

NO. OF ITERATIONS

MEAN=	.33704E+01	.38286E+01	.35315E+01
MIN=	.30000E+01	.30000E+01	.20000E+01
MAX=	.60000E+01	.70000E+01	.50000E+01
NO. OF FAILURES=	0	0	0

APPENDIX B. COMPUTER PROGRAM FOR CONFIDENCE LIMITS FOR A PROPORTION OR ERROR RATE

```
      PROGRAM CONLIM(INPUT,OUTPUT)
      COMMON/LIMITS/PL,PU,PL0,PU0,PLI,PUI,PPLI,PPUI,PLP,PUP,XL1,XL2
      COMMON/MOM/PH,QH,RHOH,XLH,N,S,R,T,ALPHA,UA,XLS
      COMMON/TAN/SL,SU,ZL,ZU,RL,RU,BL,BU,HL,HU
      COMMON/PI/S1LI,S2LI,S1UI,S2UI
      COMMON/EI/D1LI,D2LI,D1UI,D2UI,D1PLI,D2PLI,D1PUI,D2PUI
      COMMON/OUT/JT2,JT4,JT,LT2,LT4,LT
      COMMON/SMALL/XLG
    2 FORMAT(/)
    3 FORMAT(1H1)
    5 FORMAT(1X ,I2,* PERCENT CONFIDENCE INTERVAL FOR LAMBDA = (*,E12
     1.5.*,*,E12.5,*)*)
    6 FORMAT(1X ,I2,* PERCENT CONFIDENCE INTERVAL FOR P*)
    7 FORMAT(1X, *PARAMETER VALUES.....,*
     1          ,*N= *,I5,6X,*S= *,F4.0,6X,*R= *,F4.0,6X,*T= *,F2.0,6X,
     1*ALPHA=*,F4.3)
    8 FORMAT(1X,*ACCURACY VALUES.....*
     1,*MINSIG=*,I1,6X,*MINDEC=*,I1,5X,*MAXIT=*,I2)
    9 FORMAT(1X, *PARAMETER VALUES.....,*
     1          ,*N= *,I5,6X,*S= *,F4.0,6X,*R= *,F4.0,6X,*T= *,F2.0,6X,
     1*ALPHA=*,F4.3,6X,*LAMBDA= *,F4.3)
   20 FORMAT(7X,*ANDERSON-BURSTEIN =*,19X,*(*,E12.5,*,*,E12.5,*)*)
   21 FORMAT(7X,*NORMAL =*,30X,*(*,E12.5,*,*,E12.5,*)*)
   22 FORMAT(7X,*EDGEWORTH 2-TERM*,2X,*ITERATION NO.*,I2,*=*, 4X,
     1*(*,E12.5,*,*,E12.5,*)*)
   23 FORMAT(7X,*EDGEWORTH 4-TERM*,2X,*ITERATION NO.*,I2,*=*, 4X,
     1*(*,E12.5,*,*,E12.5,*)*)
   24 FORMAT(7X,*PEARSON SYSTEM  *,2X,*ITERATION NO.*,I2,* =*, 3X,*(*
     1,E12.5,*,*,E12.5,*)*)
   25 FORMAT ( 1X,*TANGENTS OF *,I2,* PERCENT CONFIDENCE REGION*)
   26 FORMAT(32X,*HORIZONTAL TANGENT*,1BX,*VERTICAL TANGENT*)
   28 FORMAT(34X,*(LIMITS FOR P)*,19X,*(LIMITS FOR LAMBDA)*)
   27 FORMAT(32X,*P*,12X,*LAMBDA*,18X,*P*,12X,*LAMBDA*)
   31 FORMAT(10X,*UPPER*,10X,2(E12.5, 3X,E12.5,10X))
   32 FORMAT(10X,*LOWER*,10X,2(E12.5, 3X,E12.5,10X))
   33 FORMAT(1X,*NO INFORMATION ON LAMBDA*)
   34 FORMAT( 1X,*USER MUST FURNISH PRIOR KNOWN VALUE OF LAMBDA. SAMPLE
     1PROVIDES NO ESTIMATE OF LAMBDA*)
   35 FORMAT(7X,*EXACT LIMITS FOR P =*18X,*(*,E12.5,*,*,E12.5,*)*)
   36 FORMAT( 1X,*REGION FOR (LAMBDA,P) IS APPROXIMATE FOR SMALL S AND
     1NOT APPLICABLE FOR S=0 OR S=1.*)
   37 FORMAT(7X,*P HAT=*,7X,E12.5)
   38 FORMAT(7X,*LAMBDA TILDE=*,E12.5)

C.....INPUT VARIABLES NEEDED TO ESTABLISH ACCURACY OF THE EDGEWORTH AND
C.....PEARSON SYSTEM APPROXIMATIONS ARE MINSIG, MINDEC, AND MAXIT.
C.....MINSIG=NUMBER OF SIGNIFICANT DIGITS OF ACCURACY FROM ITERATIONS
C.....(RELATIVE ACCURACY).
C.....MINDEC=NUMBER OF DECIMAL PLACES OF ACCURACY FROM ITERATIONS
C.....(ABSOLUTE ACCURACY).
C.....MAXIT=MAXIMUM NUMBER OF ITERATIONS (STOP EVEN IF NEITHER RELATIVE
C.....NOR ABSOLUTE ACCURACY IS ACHIEVED)
C.....NPTS=NUMBER OF POINTS USED TO DETERMINE THE CONFIDENCE REGION FOR
C.....LAMBDA AND P.
      MINSIG=3
```

```
      MAXIT=20
      MINDEC=4
      NPTS=1000

C.....BASIC INPUT VARIABLES ARE N,S,R,T, AND ALPHA (LISTED IN
C.....COMMON/MOM/)
C.....N=SAMPLE SIZE
C.....S=NUMBER OF ERRORS (S=0,1,...,N)
C.....R=NUMBER OF PAIRS OF CONSECUTIVE ERRORS (R=0,1,...,S-1)
C.....T=NUMBER OF ERRORS IN FIRST AND LAST TRIALS (T=0,1,2)
C.....100*(1-2*ALPHA)=PERCENTAGE CONFIDENCE INTERVAL.
C.....BECAUSE IT USES TABLES INSTEAD OF EQUATIONS, THE PEARSON SYSTEM
C.....(SUBROUTINE PSA) CAN HANDLE ONLY ALPHA=.025 AND .050.
C.....XLG=A VALUE OF LAMBDA THAT MUST BE SUPPLIED BY THE USER IF S=0 OR
C.....S=1.
      N=20000 $ R=13. $ S=38. $ T=0. $ ALPHA=.05

      CALL DEPEND(MINSIG,MINDEC,MAXIT,NPTS)

C.....PRINT THE INPUT DATA AND ACCURACY VALUES
      PRINT3
      PRINT2
      IF(S.LE.1.)PRINT9,N,S,R,T,ALPHA,XLG
      IF(S.GT.1.)PRINT7,N,S,R,T,ALPHA
      PRINT8,MINSIG,MINDEC,MAXIT
      PRINT2 $ PRINT2

C.....PRINT THE RESULTS
      IF(S.LE.1.)GO TO 30
      IA=INT(ALPHA*1000.) $ II=100-2*IA/10
      PRINT25,II
      PRINT2
      PRINT26
      PRINT 28
      PRINT2
      PRINT27
      PRINT31,SU,ZU,RU,BU
      PRINT32,SL,ZL,RL,BL
   30 IF(S.LE.1.)PRINT36
      PRINT2 $ PRINT2
      IF(S.GT.1.)PRINT5,II,XL1,XL2
      IF(S.GT.1.)GO TO 41
      PRINT33
      PRINT2
      PRINT34
      PRINT2
      PRINT35,PL,PU
      CALL EXIT
   41 PRINT2
      PRINT2
      PRINT2
      PRINT6,II
      PRINT20,   PL,PU
      PRINT2
      PRINT21,   PL0,PU0
      PRINT2
      PRINT22,JT2,D1LI,D1UI
      PRINT22,LT2,D2LI,D2UI
      PRINT2
      PRINT23,JT4,D1PLI,D1PUI
      PRINT23,LT4,D2PLI,D2PUI
      PRINT2
      PRINT24,JT,S1LI,S1UI
      PRINT24,LT,S2LI,S2UI
      PRINT2 $ PRINT2
      PRINT37,PH
      PRINT38,XLH
      END
```

```
      SUBROUTINE DEPEND(MINSIG,MINDEC,MAXIT,NPTS)
      COMMON/LIMITS/PL,PU,PLO,PUO,PLI,PUI,PPLI,PPUI,PLP,PUP,XL1,XL2
      COMMON/MJM/PH,QH,RHOH,XLH,N,S,R,T,ALPHA,UA,XLS

C.....SUBROUTINE PARAM DEFINES SOME BASIC PARAMETERS.

      CALL PARAM

C.....SUBROUTINE REGION DEFINES THE CONFIDENCE REGION FOR (LAMBDA,P),
C.....THE TWO VERTICAL TANGENTS, AND THE TWO HORIZONTAL TANGENTS.

      CALL REGION(NPTS)


C.....SUBROUTINE PLOT PLOTS THE CONFIDENCE REGION ON MICROFILM.

      CALL PLOT
      IF(S.LE.1.)GO TO 10

C.....SUBROUTINE LIMLAM DEFINES THE CONFIDENCE LIMITS FOR LAMBDA.

      CALL LIMLAM

C.....SUBROUTINE NORMAL DETERMINES THE NORMAL APPROXIMATION 1-2*ALPHA
C.....CONFIDENCE LIMITS PLO AND PUO. IT UTILIZES THE 100*ALPHA PERCENTAG
C.....E POINT OF THE STANDARD NORMAL DISTRIBUTION (EQ. 2.13).

      CALL NORMAL(PLO,PUO)

C.....SUBROUTINE EDGEW DETERMINES BOTH THE 2-TERM AND THE 4-TERM EDGEWOR
C.....TH APPROXIMATION. THE LIMITS, DETERMINED BY ITERATION, ARE PLI AND
C.....PUI FOR THE 2-TERM CASE AND PPLI AND PPUI FOR THE 4-TERM CASE.

      CALL EDGEW(PLO,PUO,PLI,PUI,PPLI,PPUI,IT2,IT4,MINDEC,MAXIT,MINSIG)

C.....SUBROUTINE PSA DETERMINES THE PEARSON SYSTEM APPROXIMATION.
C.....THE LIMITS, DETERMINED BY ITERATION, ARE PLP AND PUP.

      CALL PSA(PLO,PUO,PLP,PUP,IT,MINDEC,MAXIT,MINSIG)

C.....SUBROUTINE ANDBUR IMPROVES ON THE POISSON APPROXIMATION FOR
C.....BINOMIAL CONFIDENCE LIMITS.
C.....PLIND AND PUIND ARE THE ANDERSON-BURSTEIN IMPROVED LIMITS FOR IND-
C.....EPENDENT TRIALS. PL AND PU ARE THE MODIFIED ANDERSON-BURSTEIN LIMI
C.....TS FOR DEPENDENT TRIALS.

   10 CALL ANDBUR(PL,PU,PLIND,PUIND,MINSIG,MINDEC)

      RETURN
      END
```

```
      SUBROUTINE PARAM
      COMMON/MJM/PH,QH,RHOH,XLH,N,S,R,T,ALPHA,UA,XLS
      DIMENSION D3(12),D8(12)
      DATA(D3=.103,.088,.080,.069,.060,.052,.046,.041,.035,.032,.029,.02
     16)
      DATA(D8=.197,.174,.163,.152,.145,.136,.128,.122,.115,.109,.098,.08
     18)

C.....DEFINE UA, THE UPPER 100*ALPHA PERCENTAGE PT. OF THE STANDARDIZED
C.....NORMAL DISTRIBUTION. (EQ. 2.9). SUBROUTINE MDNRIS(1.-ALPHA,UA,IER)
C.....IS A SUBROUTINE FROM IMSL. IT COMPUTES
C.....THE INVERSE GAUSSIAN INTEGRAL. FOR INSTANCE, IF 1-ALPHA=.95 IS THE
C.....INPUT VALUE, THEN  UA=1.64485 IS THE OUTPUT VALUE. IER IS AN ERROR
C.....INDICATOR. (NOTE.. IMSL=INTERNATIONAL MATHEMATICAL AND STATISTICAL
C.....LIBRARIES).

      CALL MDNRIS(1.-ALPHA,UA,IER)

C.....DEFINE P HAT (EQ. 2.1)
      PH=S/N

C.....DEFINE LAMBDA HAT (EQ.2.2)
      QH=1.-PH
      AH=S-PH
      BH=QH*(2.*S-T)
      WA=(R-BH+AH)*(R-BH+AH)+4.*R*(1.-2.*PH)*AH
      IF(S.EQ.0.)XLH=0.
      IF(S.NE.0.)XLH=(R-BH+AH+SQRT(WA))/(2.*AH)


C.....DEFINE LAMBDA TILDE
      IERROR=INT(S+1.E-5)
      IF(IERROR.LT.13)Q3=D3(IERROR)
      IF(IERROR.LT.13)Q8=D8(IERROR)
      IF(IERROR.GE.13)Q3=1.043*S**(-1.442)
      IF(IERROR.GE.13)Q8=8.65*S**(-1.824)
      XLT=(XLH+1.6*Q3-0.6*Q8)/( 1.-2.*(Q8-Q3))
      IF((XLH.GE.0.).AND.(XLH.LE..3-Q3))XLT=.3*XLH/(.3-Q3)
      IF((XLH.GE..8-Q8).AND.(XLH.LE.1.))XLT=(.2*XLH+Q8)/(.2+Q8)
      IF(XLT.LT.0.)XLT=0.
      IF(XLT.GT.1.)XLT=1.
      XLH=XLT

C.....DEFINE LAMBDA STAR (EQ.2.4)
      IF(S.GT.0.)XLS=N*R/((N-1.)*S)

C.....DEFINE RHO HAT (EQ.2.8)
      RHOH=(XLH-PH)/QH
      RETURN
      END

*************************************************************************
```

```
      SUBROUTINE LIMLAM
      COMMON/MJM/PH,QH,RHOH,XLH,N,S,R,T,ALPHA,UA,XLS
      COMMON/LIMITS/PL,PU,PLO,PUO,PLI,PUI,PPLI,PPUI,PLP,PUP,XL1,XL2

C.....DEFINE NORMAL CONFIDENCE LIMITS FOR LAMBDA (SEE EQ. 2.17)
      YI=UA*UA+2.*S*XLH $ YJ=4.*S*XLH*XLH*(UA*UA+S)
      XL1=(YI-SQRT(YI*YI-YJ))/(2.*(UA*UA+S))
      XL2=(YI+SQRT(YI*YI-YJ))/(2.*(UA*UA+S))
      RETURN
      END

*************************************************************************
```

```
      SUBROUTINE REGION(KZ)

C.....THIS SUBROUTINE DEFINES THE BOUNDARY (EQ. 2.16) OF THE CONFIDENCE
C.....REGION FOR (LAMBDA, P). IT ALSO LOCATES THE TWO VERTICAL TANGENTS
C.....AND THE TWO HORIZONTAL TANGENTS.
      COMMON/NUM/PH,QH,RHOH,XLH,N,S,R,T,ALPHA,UA,XLS
      COMMON/PLT/ GL(500),GU(500),GP(500),KE
      COMMON/TAN/SL,SU,ZL,ZU,RL,RU,BL,BU,HL,HU

C.....DETERMINE XC, THE VALUE OF THE UPPER 200*ALPHA PERCENTAGE POINT OF
C.....THE CHI-SQUARED DISTRIBUTION WITH TWO DEGREES OF FREEDOM.
C.....MDCHI IS OBTAINED FROM THE INTERNATIONAL MATHEMATICAL AND STATISTI
C.....CAL LIBRARIES,INC.

      CALL MDCHI(1.-2.*ALPHA,2.,XC,IER)

C.....FIND THE INTERVAL FOR LAMBDA WITHIN .1 ON EACH END.
      MU=0
      IZ=11
      DO 100 I=1,IZ
      XRT=0.
      YL=(I-1.)/(IZ-1.)
      CALL CURVES(YL,XLH,QH  ,PH,XC,N,PL,PU,XRT)
      IF((XRT.EQ.YL).AND.(YL.LT.XLH))HL=YL
      IF((XRT.EQ.YL).AND.(YL.GT.XLH))HU=YL
      IF((XRT.EQ.YL).AND.(YL.GT.XLH))GO TO 101
  100 CONTINUE
  101 CONTINUE

C.....DEFINE THE BOUNDARY OF THE CONFIDENCE REGION AND LOCATE THE POINTS
C.....OF TANGENCY
      KD=0
      KE=1
      NL=NU=0

C.....KX=NUMBER OF POINTS TO BE PLOTTED
      KX=100

C.....KZ=NUMBER OF POINTS USED TO DEFINE BOUNDARY AND TANGENTS
      KT=KZ/KX
      DO 333 I=1,KZ
      YL=(KZ-I)*HL/(KZ-1.)+(I-1.)*HU/(KZ-1.)
      TL=PL $ TU=PU
      CALL CURVES(YL,XLH,QH  ,PH,XC,N,PL,PU,XRT)
   17 FORMAT(1X,I3,3X,3(E12.5,3X))

C.....FIND THE COORDINATES (BL,RL) AND (BU,RU) OF VERTICAL TANGENTS
      IF((TU.EQ.-1.E6).AND.(PU.NE.-1.E6))BL=YL
      IF((TU.EQ.-1.E6).AND.(PU.NE.-1.E6))RL=(PL+PU)/2.
      IF((TL.NE.-1.E6).AND.(PL.EQ.-1.E6))BU=WL
      IF((TL.NE.-1.E6).AND.(PL.EQ.-1.E6))RU=(TL+TU)/2.

C.....FIND THE COORDINATES (ZL,SL) AND (ZU,SU) OF HORIZONTAL TANGENTS
      IF((PL*TL.GT.0.).AND.(PL.GT.TL))NL=NL+1
      IF(NL.EQ.1)SL=PL
```

```
      IF(NL.EQ.1)ZL=(YL+WL)/2.
      IF((PL*TL.GT.0.).AND.(PU.LT.TU))NU=NU+1
      IF(NU.EQ.1)SU=PU
      IF(NU.EQ.1)ZU=(YL+WL)/2.
      WL=YL
      IF(PL.EQ.-1.E6)GO TO 333
      KD=KD+1
      IF(KD.EQ.(KD/KT)*KT)KE=KE+1

C.....GL=LOWER BOUNDARY
C.....GU=UPPER BOUNDARY
C.....GP=VALUES OF LAMBDA (ABSCISSA)
      GL(KE)=PL $ GU(KE)=PU $ GP(KE)=YL
  333 CONTINUE
      RETURN
      END

*************************************************************************
```

```
      SUBROUTINE CURVES (A,B,C,D,E,N,PL,PU,XRT)

C.....THIS SUBROUTINE COMPUTES THE BOUNDARY OF THE CONFIDENCE REGION
C.....FOR LAMBDA AND P. (EQ. 2.16).
      AQ(A,B,C,D)=  A*(1.-A)*(1.-A)+2.*C*A*(1.-A)*(B-A)+C*(1.-2.*D+A)*
     1(B-A)*(B-A)
      BQ(A,B,C,D,E,N)=-A*(1.-A)*(2.*D*(1.-A)+2.*C*D*(B-A)+E*C*(1.-2.*D+A
     1)/N)
      CQ(A,D)=D*D*A*(1.-A)*(1.-A)
      YQ1(AA,BB,CC)=(-BB+SQRT(BB*BB-4.*AA*CC))/(2.*AA)
      YQ2(AA,BB,CC)=(-BB-SQRT(BB*BB-4.*AA*CC))/(2.*AA)
    7 FORMAT(1X,6(E12.5,3X))
      IF(B .EQ.0.)13,14
   13 AA=(1.-A)*(1.-A)+2.*C*(1.-A)*(1.-A)+A*C*(1.-2.*D+A)
      BB=-(1.-A)*(2.*D*(1.-A)+2.*C*D*(B-A)+E*C*(1.-2.*D+A)/N)
      CC=D*D*(1.-A)*(1.-A)
      GO TO 15
   14  AA=AQ(A,B,C,D)   $ BB=BQ(A,B,C,D,E,N)   $ CC=CQ(A,D)
   15 IF(BB*BB-4.*AA*CC.LT.0.)PL=PU=-1.E6
      IF(A.EQ.1.)PL=PU=-1.E6
      IF(A.EQ.1.)GO TO 10
      IF(BB*BB-4.*AA*CC.LT.0.)XRT=A
      IF(BB*BB-4.*AA*CC.LT.0.)GO TO 10
      IF(AA.EQ.0.)GO TO 10
      PL=YQ2(AA,BB,CC) $ PU=YQ1(AA,BB,CC)
   10 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE PLOT

C.....THIS SUBROUTINE PLOTS THE CONFIDENCE REGION FOR (LAMBDA, P)
      COMMON/MAM/PH,QH,RHOH,XLH,N,S,R,T,ALPHA,UA,XLS
      COMMON/DDSCALE/XMIN,XMAX,YMIN,YMAX,MINX,MAXX,MINY,MAXY,SCX,SCY,
     1NSCX,NSCY,MSCX,MSCY,ISCX,ISCY
      COMMON/DD/IN,IOR,IT,IS,IC,ICC,IX,IY
      COMMON/TAN/SL,SU,ZL,ZU,RL,RU,BL,BU,HL,HU
      COMMON/PLT/ GL(500),GU(500),GP(500),KE
      DIMENSION AB(2),AC(2),AD(2),AE(2)
      CALL DDINIT(3,18HM. J. MILES, X3506)
      MINX=MINY=0 $ MAXX=MAXY=1000
      CALL DDBOX(MINX,MAXX,MINY,MAXY)

C.....SIZE THE ORDINATE FROM ZERO TO THE LARGEST POWER OF .1 THAT INCLUD
C.....ES THE REGION. DRAW TEN EQUALLY SPACED TIC MARKS.
      GG=.1
      DO 27 I=1,10
      GG=GG*10.
      K1=INT(SU*GG)
      IF(K1.GT.1.)GO TO 28
   27 CONTINUE
   28 SM=1.*(I-2.)
      YMAX=1.*10.**(-SM) $ YMIN=0.

C.....SIZE THE ABSCISSA FROM THE LARGEST TO THE SMALLEST INTEGER
C.....MULTIPLE OF .1 THAT INCLUDES THE REGION. DRAW TIC MARKS AT EVERY
C..... .1.
      JX1=INT(HL*10.+1.E-6) $ JX2=INT(HU*10.+1.E-6)
      XMIN=.1*JX1 $ XMAX=.1*JX2
      CALL DDPT
      KX=JX2-JX1 $ KY=10
      MX=MAXX/KX
      MY=MAXY/KY
      LX=KX-1 $ LY=KY-1

C.....DRAW THE TIC MARKS ON THE ABSCISSA FOR LAMBDA AT EVERY .1.
      DO 10 I=1,LX
      IX=I*MX
      IY=0
      CALL DDBP
      IX=I*MX
      IY=20
      CALL DDVC
      IX=I*MX
      IY=1000
      CALL DDBP
      IY=980
      IX=I*MX
      CALL DDVC
   10 CONTINUE

C.....DRAW 10 EQUALLY SPACED TIC MARKS ON THE ORDINATE FOR P.
      DO 20 I=1,LY
      IY=I*MY
```

```
      IX=0
      CALL DDBP
      IY=I*MY
      IX=20
      CALL DDVC
      IY=I*MY
      IX=930
      CALL DDBP
      IY=I*MY
      IX=MAXX
      CALL DDVC
   20 CONTINUE

C.....DRAW THE UPPER AND LOWER BOUNDARY OF THE REGION.
      CALL DDGRAPH(KE,GP,GU,1,1,1)
      CALL DDGRAPH(KE,GP,GL,1,1,1)

C.....CONNECT THE UPPER AND LOWER CURVES TO COMPLETE THE BOUNDARY
      AB(1)=GL(1) $ AB(2)=GU(1)$AC(1)=GL(KE) $ AC(2)=GU(KE)
      AD(1)=GP(1) $ AD(2)=GP(1) $ AE(1)=GP(KE) $ AE(2)=GP(KE)
      CALL DDGRAPH(2,AD,AB,1,1,1)
      CALL DDGRAPH(2,AE,AC,1,1,1)
      CALL DDFRAME
      RETURN
      END
```

****

```
      SUBROUTINE NORMAL(PLO,PUO)
      COMMON/MJM/PH,QH,RHOH,XLH,N,S,R,T,ALPHA,UA,XLS

C.....PLO AND PUO ARE THE NORMAL 1-2*ALPHA CONFIDENCE LIMITS (SEE EQ.
C..... 2.13)
      RHOA=ABS(RHOH)  $ G=1. $ IF(N.EQ.(N/2)*2)G=RHOA/RHOH
      V=N*QH*(1.-RHOH*RHOH-2.*RHOH/N+2.*G*(RHOA**(N+1.))/N)/((1.-RHOH)*
     1(1.-RHOH))
      W = (2. * S + 1.) * N
      PUO = (V * UA * UA + W + SQRT ((V * UA * UA + W) * (V * UA * UA +
     1W) - W * W)) / (2. * N * N)
      Z = (2. * S - 1.) * N
      PLO=(V*UA*UA+Z-SQRT((V*UA*UA+Z)*(V*UA*UA+Z)-Z*Z))/(2.*N*N)
      IF(PUO.GT.1.)PUO=1.
      RETURN
      END

*****************************************************************************
```

```
      SUBROUTINE EDGEW(PL0,PU0,PLI,PUI,PPLI,PPUI,IT2,IT4,L,IZ,ISIG)

C.....THIS SUBROUTINE DETERMINES BOTH THE EDGEWORTH 2 AND 4 TERM
C.....APPROXIMATIONS
      COMMON/OUT/JT2,JT4,JT,LT2,LT4,LT
      COMMON/EI/D1LI,D2LI,D1UI,D2UI,D1PLI,D2PLI,D1PUI,D2PUI
      COMMON/MJM/PH,QH,RHOH,XLH,N,S,R,T,ALPHA,UA,XLS
      PHI(X)=EXP(-X*X/2.)/SQRT(2.*3.141592654)
      P3(X)=(EXP(-X*X/2.)/SQRT(2.*3.141592654))*(-X**3+3.*X)
      P5(X)=(EXP(-X*X/2.)/SQRT(2.*3.141592654))*(-X**5+10.*X**3-15.*X)
      J2=J4=0
      NALP=NALPP=0
      JALP=JALPP=0
      ML=MU=MPL=MPU=0
      I=0
      IT2=IT4=0
      UAL=UAU=UA
      UPAL=UPAU=UA
      HLI=HUI=ALPHA
      PHL=PL0 $ PHU=PU0
      PLI=PL0 $ PUI=PU0
      PPLI=PL0 $ PPUI=PU0
      QHL=1.-PLI $ QHU=1.-PUI

C.....L=NUMBER OF DECIMAL PLACES DESIRED FOR AGREEMENT ON CONSECUTIVE
C.....ITERATIONS.
      FT=1.*10**(-L)
      IF((PL0.LE.0.).OR.(PU0.GE.1.))GO TO 30

C.....BEGIN ITERATION. ITERATION LOOPS BACK TO STATEMENT 35 IF LIMITS
C. ...FROM TWO CONSECUTIVE ITERATIONS ARE NOT SUFFICIENTLY CLOSE.
   35 IF(I.EQ.IZ)GO TO 30
      RHLH=(XLH-PLI)/QHL
      RHUH=(XLH-PUI)/QHU

C.....DEFINE PLI (EQ. 3.11)
      CLI=PLI
      RHLA=ABS(RHLH) $ G=1. $ IF(N.EQ.(N/2)*2)G=RHLA/RHLH
      VHL=N*QHL*(1.-RHLH*RHLH-2.*RHLH/N+2.*G*(RHLA**(N+1.))/N)/((1.
     1-RHLH)*(1.-RHLH))
      GL=1.-RHLH $ DL=GL*GL*GL $ HL=VHL*VHL
      BHL=QHL*(1.-2.*PHL)*(N+(6.*RHLH/(DL))*(N-1.-(N+1.)*RHLH))/(6.*
     1(HL**.75))
      CHL=(1.-6.*PHL*QHL)*(1.+10.*RHLH+RHLH*RHLH)/(24.*N*QHL*(1.-RHLH*
     1RHLH))
      IF(PLI.GT.0.)ALI=ALPHA-(BHL/SQRT(PLI))*(UAL*UAL-1.)*PHI(UAL)
      IF(((ALI.LE.0.)).OR.((ALI.GT.0.5)))NALP=1

C.....ONCE ALI IS LESS THAN ZERO SET PLI=0 AND HOLD IT THERE REGARDLESS
C.....OF THE VALUE ALI   ASSUMES DURING ITERATION. OBSERVE THE SAME RULE
C.....FOR ALPI.
      IF(NALP.EQ.1)PLI=0.
      IF((ALI.GT.0.).AND.(ALI.LT..5))CALL MDNRIS(1.-ALI,UAL,IER)
      CL=(2.*S-1.)*N
      BL=VHL*UAL*UAL+CL
```

```
      IF(PLI.NE.0.)PLI=(BL-SQRT(BL*BL-CL*CL))/(2.*N*N)
      QHL=1.-PLI
      PHL=PLI

C.....DEFINE PUI (EQ. 3.11)
      CUI=PUI
      RHUA=ABS(RHUH) $ G=1. $ IF(N.EQ.(N/2)*2)G=RHUA/RHUH
      VHU=N*QHU*(1.-RHUH*RHUH-2.*RHUH/N+2.*G*(RHUA**(N+1.))/N)/((1.
     1-RHUH)*(1.-RHUH))
      GU=1.-RHUH $ DU=GU*GU*GU $ HU=VHU*VHU
      BHU=QHU*(1.-2.*PHU)*(N+(6.*RHUH/(DU))*(N-1.-(N+1.)*RHUH))/(6.*
     1(HU**.75))
      CHU=(1.-6.*PHU*QHU)*(1.+10.*RHUH+RHUH*RHUH)/(24.*N*QHU*(1.-RHUH*
     1RHUH))
      AUI=ALPHA+(BHU/SQRT(PUI))*(UAU*UAU-1.)*PHI(UAU)
      IF((AUI.LT.0.).OR.(AUI.GT.0.5))JALP=1
      IF(JALP.EQ.1)PUI=1.
      IF((AUI.GT.0.).AND.(AUI.LT..5))CALL MDNRIS(1.-AUI,UAU,IER)
      CU=(2.*S+1.)*N
      BU=VHU*UAU*UAU+CU
      PUI=(BU+SQRT(BU*BU-CU*CU))/(2.*N*N)
      QHU=1.-PUI
      PHU=PUI

C.....DEFINE PPLI
      CPLI=PPLI

C.....APLI IS DEFINED IN EQ. 4.4
      IF(NALPP.NE.1)HLI=ALPHA-BHL*(UPAL*UPAL-1.)*PHI(UPAL)/SQRT(PPLI)
      IF(PPLI.GT.0.)APLI=HLI+(CHL*P3(UPAL)+0.5*BHL*BHL*P5(UPAL))/PPLI
      IF(((APLI.LE.0.)).OR.(APLI.GT.0.5))NALPP=1
      IF((APLI.GT.0.).AND.(APLI.LT..5))CALL MDNRIS(1.-APLI,UPAL,IER)
      BPL=VHL*UPAL*UPAL+CL
      IF(PPLI.NE.0.)PPLI=(BPL-SQRT(BPL*BPL-CL*CL))/(2.*N*N)
      IF(NALPP.EQ.1)PPLI=0.

C.....DEFINE PPUI
      CPUI=PPUI

C.....APUI IS DEFINED IN EQ. 4.4
      HUI=ALPHA+BHU*(UPAU*UPAU-1.)*PHI(UPAU)/SQRT(PPUI)
      APUI=HUI-(CHU*P3(UPAU)+0.5*BHU*BHU*P5(UPAU))/PPUI
      IF((APUI.LT.0.).OR.(APUI.GT.0.5))JALPP=1
      IF(JALPP.EQ.1)PPUI=1.
      IF((APUI.GT.0.).AND.(APUI.LT..5))CALL MDNRIS(1.-APUI,UPAU,IER)
      BPU=VHU*UPAU*UPAU+CU
      PPUI=(BPU+SQRT(BPU*BPU-CU*CU))/(2.*N*N)
      I=I+1

C.....TEST THE ACCURACY OF THE LIMITS.
      CALL SIGFIG(ISIG,JOK,PLI,PUI,CLI,CUI,KPL,KPU,KCL,KCU)
      CALL SIGFIG(ISIG,JOK,PPLI,PPUI,CPLI,CPUI,KPPL,KPPU,KCPL,KCPU)
      FL=ABS(PLI-CLI) $ FPL=ABS(PPLI-CPLI)
      FU=ABS(PUI-CUI) $ FPU=ABS(PPUI-CPUI)

C.....IF THE LIMITS AGREE SUFFICIENTLY FOR TWO CONSECUTIVE ITERATIONS,
```

```
C.....SET ML=1, ETC.
      IF((FL.LT.FT).OR.(KPL.EQ.KCL))ML=1
      IF((FU.LT.FT).OR.(KPU.EQ.KCU))MU=1

C.....IF BOTH UPPER AND LOWER LIMITS AGREE SUFFICIENTLY WITH THEIR PREVI
C.....OUS VALUES, NOTE THE NUMBER OF THE ITERATION.
      IF((ML*MU.EQ.1).AND.(J2.EQ.0))IT2=I
      IF(ML*MU.EQ.1)J2=1
      IF((FPL.LT.FT).OR.(KPPL.EQ.KCPL))MPL=1
      IF((FPU.LT.FT).OR.(KPPU.EQ.KCPU))MPU=1
      IF((MPL*MPU.EQ.1).AND.(J4.EQ.0))IT4=I
      IF(MPL*MPU.EQ.1)J4=1
      IF((ML*MU.NE.1).AND.(I.EQ.IZ))IT2=I
      IF((MPL*MPU.NE.1).AND.(I.EQ.IZ))IT4=I

C.....TEST THE NEED FOR ADDITIONAL ITERATIONS.
      IF(ML*MU*MPL*MPU.EQ.1)30,35
   30 CONTINUE
      D1LI=CLI  $ D2LI=PLI
      D1UI=CUI  $ D2UI=PUI
      D1PLI=CPLI $ D2PLI=PPLI
      D1PUI=CPUI $ D2PUI=PPUI
      IF((PLI.LE.0.))PLI=0.
      IF(PUI.GE.1.)PUI=1.
      IF((PPLI.LE.0.))PPLI=0.
      IF(PPUI.GE.1.)PPUI=1.
   73 CONTINUE
      JT2=IT2-1 $ JT4=IT4-1
      IT2=IT2 $ LT4=IT4
      RETURN
      END
```

****

```
      SUBROUTINE SIGFIG(JSIG,JOK,PLI,PUI,CLI,CUI,KPL,KPU,KCL,KCU)

C.....THIS SUBROUTINE TESTS TO SEE THAT TWO CONSECUTIVE ITERATIVE
C.....VALUES DIFFER BY LESS THAN JSIG SIGNIFICANT DIGITS.
C.....ANY VALUES OF PLI OR CLI CLOSE TO ZERO ARE SET EQUAL TO ZERO.
C.....THIS LOOP IS SATISFIED WHEN THE VALUES ARE MULTIPLIED BY THE
C.....POWER OF TEN THAT RENDERS THE VALUE EQUAL TO AN INTEGER THAT HAS
C.....JSIG DIGITS.
      JOK=0
      KPL=KPU=KCL=KCU=0
      JJ=10**(JSIG-1)
      JA=15
      IF(PLI.LT.1.*10**(JSIG-JA))PLI=0.
      IF(CLI.LT.1.*10**(JSIG-JA))CLI=0.
      DO 20 J=1,JA
      IF(PLI.NE.0.)JPL=INT(PLI*(10**J))
      IF(PLI.EQ.0.)JPL=JJ
      IF(CLI.NE.0.)JCL=INT(CLI*(10**J))
      IF(CLI.EQ.0.)JCL=JJ
      JPU=INT(PUI*(10**J))
      JCU=INT(CUI*(10**J))
      IF((JPL.GE.JJ).AND.(JPL.LT.10*JJ))KPL=JPL
      IF((JPU.GE.JJ).AND.(JPU.LT.10*JJ))KPU=JPU
      IF((JCL.GE.JJ).AND.(JCL.LT.10*JJ))KCL=JCL
      IF((JCU.GE.JJ).AND.(JCU.LT.10*JJ))KCU=JCU
      IF(KPL*KPU*KCL*KCU.NE.0)GO TO 25
      IF(KPU*KCU.NE.0)GO TO 25
   20 CONTINUE
   25 IF(J.LE.15)JOK=1
      RETURN
      END
```

****

```
      SUBROUTINE PSA(PL0,PU0,PLI,PUI,IT,LL,IZ,ISIG)
      COMMON/PQ/B2L(21,68),B2U(21,68),B5L(21,68),B5U(21,68)
      COMMON/MJM/PH,QH,RHOH,XLH,N,S,R,T,ALPHA,UA,XLS
      COMMON/PI/S1LI,S2LI,S1UI,S2UI
      COMMON/OUT/JT2,JT4,JT,LT2,LT4,LT
      DIMENSION BETA1(21),BETA2(65)
   11 FORMAT(1X,*ALPHA .NE. .025*)

C.....SUBROUTINE PEART ENTERS THE PEARSON AND HARTLEY TABLE 32 FOR
C.....ALPHA=.025,.050.

      CALL PEART

C.....LL=NUMBER OF DECIMAL PLACES DESIRED FOR AGREEMENT ON CONSECUTIVE
C.....ITERATIONS.
      FT=1.*10**(-LL)

C.....DEFINE BETA1
      DO 77 I=1,21
      BETA1(I)= (.1*(I-1.))**2
   77 CONTINUE

C.....DEFINE BETA2
      DO 75 I=1,65
      BETA2(I)=1.6+0.2*(I-1.)
   75 CONTINUE
      IT=0
      ML=MU=0
      M=0
      PLI=PL0 $ PUI=PU0
      PHL=PL0 $ PHU=PU0
      IF(PL0.EQ.0.)GOTO 33
      IF(PU0.EQ.1.)GO TO 33

C.....ITERATION WILL CAUSE RETURN TO STATEMENT 35
   35 CONTINUE
      IF(M.EQ.IZ)GO TO 33
      M=M+1
      QHL=1.-PLI $ QHU=1.-PUI
      RHLH=(XLH-PLI)/QHL
      RHUH=(XLH-PUI)/QHU
      IF((RHLH.GE. 1.).OR.(RHUH.GE. 1.))33,72
   72 IF((RHLH.LE.-1.).OR.(RHUH.LE.-1.))33,74
   74 VHL=N*QHL*(1.-RHLH*RHLH-2.*RHLH/N)/((1.-RHLH)*(1.-RHLH))
      GL=1.-RHLH $ DL=GL*GL*GL $ HL=VHL*VHL
      VHU=N*QHU*(1.-RHUH*RHUH-2.*RHUH/N)/((1.-RHUH)*(1.-RHUH))
      GU=1.-RHUH $ DU=GU*GU*GU $ HU=VHU*VHU
      BHL=QHL*(1.-2.*PHL)*(N+(6.*RHLH/(DL))*(N-1.-(N+1.)*RHLH))/(6.*
     1(HL**.75))
      BHU=QHU*(1.-2.*PHU)*(N+(6.*RHUH/(DU))*(N-1.-(N+1.)*RHUH))/(6.*
     1(HU**.75))
      CHL=(1.-6.*PHL*QHL)*(1.+10.*RHLH+RHLH*RHLH)/(24.*N*QHL*(1.-RHLH*
     1RHLH))
      CHU=(1.-6.*PHU*QHU)*(1.+10.*RHUH+RHUH*RHUH)/(24.*N*QHU*(1.-RHUH*
     1RHUH))
```

```
      IF(PLI.EQ.0.)GO TO 33

C.....B1,B2,C1, AND C2 ARE EQUATIONS 5.1
      B1=36.*BHL*BHL/PLI
      B2=3.+24.*CHL/PLI
      C1=36.*BHU*BHU/PUI
      C2=3.+24.*CHU/PUI
      K=J=0

C.....LOCATE B1 IN THE TABLE
      DO 30 I=1,20
      IF((B1.GE.BETA1(I)).AND.(B1.LE.BETA1(I+1)))J=I
      IF(J.GT 0)GO TO 40
   30 CONTINUE
   40 CONTINUE

C.....LOCATE B2 IN THE TABLE.
      DO 88 I=1,64
      IF((B2.GE.BETA2(I)).AND.(B2.LE.BETA2(I+1)))K=I
      IF(K.GT.0)GO TO 92
   88 CONTINUE
   92 CONTINUE

C.....LOCATE C1 IN THE TABLE
      I=L=0
      DO 50 J1=1,20
      IF((C1.GE.BETA1(J1)).AND.(C1.LE.BETA1(J1+1)))I=J1
      IF(I.GT.0)GO TO 60
   50 CONTINUE
   60 CONTINUE

C.....LOCATE C2 IN THE TABLE.
      DO 89 J1=1,64
      IF((C2.GE.BETA2(J1)).AND.(C2.LE.BETA2(J1+1)))L=J1
      IF(L.GT.0)GO TO 84
   89 CONTINUE
   84 CONTINUE

C.....WHENEVER B1,B2,C1, OR C2 IS OFF A TABLE, ASSIGN IT THE NEAREST
C.....TABULAR VALUE
      IF((K.EQ.0).AND.(B2.LT.BETA2(1)))K=1
      IF((K.EQ.0).AND.(B2.LT.BETA2(1)))B2=BETA2(1)
      IF((K.EQ.0).AND.(B2.GT.BETA2(65)))K=64
      IF((K.EQ.0).AND.(B2.GT.BETA2(65)))B2=BETA2(65)
      IF((J.EQ.0).AND.(B1.LT.BETA1(1)))J=1
      IF((J.EQ.0).AND.(B1.LT.BETA1(1)))B1=BETA1(1)
      IF((J.EQ.0).AND.(B1.GT.BETA1(21)))J=20
      IF((J.EQ.0).AND.(B1.GT.BETA1(21)))B1=BETA1(21)
      IF((L.EQ.0).AND.(C2.LT.BETA2(1)))L=1
      IF((L.EQ.0).AND.(C2.LT.BETA2(1)))C2=BETA2(1)
      IF((L.EQ.0).AND.(C2.GT.BETA2(65)))L=64
      IF((L.EQ.0).AND.(C2.GT.BETA2(65)))C2=BETA2(65)
      IF((I.EQ.0).AND.(C1.LT.BETA1(1)))I=1
      IF((I.EQ.0).AND.(C1.LT.BETA1(1)))C1=BETA1(1)
      IF((I.EQ.0).AND.(C1.GT.BETA1(21)))I=20
      IF((I.EQ.0).AND.(C1.GT.BETA1(21)))C1=BETA1(21)
```

```
C.....USE DOUBLE INTERPOLATION ON TABLE FOR LOWER LIMIT
      IF(ALPHA.EQ..05)78,79
   78 UL1=((B2-BETA2(K))*B5U(J,K+1)+(BETA2(K+1)-B2)*B5U(J,K))/(BETA2(K+1
     1)-BETA2(K))
      UL2=((B2-BETA2(K))*B5U(J+1,K+1)+(BETA2(K+1)-B2)*B5U(J+1,K))/(BETA2
     1(K+1)-BETA2(K))
      GO TO 80
   79 IF(ALPHA.NE..025)PRINT11
      IF(ALPHA.NE..025)CALL EXIT
      UL1=((B2-BETA2(K))*B2U(J,K+1)+(BETA2(K+1)-B2)*B2U(J,K))/(BETA2(K+1
     1)-BETA2(K))
      UL2=((B2-BETA2(K))*B2U(J+1,K+1)+(BETA2(K+1)-B2)*B2U(J+1,K))/(BETA2
     1(K+1)-BETA2(K))
   80 IF(J.LT.19)UL=((B1-BETA1(J))*UL2+(BETA1(J+1)-B1)*UL1)/(BETA1(J+1)
     1-BETA1(J))
      IF(J.GE.19)UL=UL2

C.....DETERMINE PLI (EQS.3.11)
      CLI=PLI
      CL=(2.*S-1.)*N
      BL=VHL*UL*UL+CL
      PLI=(BL-SQRT(BL*BL-   CL*CL))/(2.*N*N)

C.....USE DOUBLE INTERPOLATION ON TABLE FOR UPPER LIMIT
      IF(ALPHA.EQ..05)81,82
   81 UU1=((C2-BETA2(L))*B5L(I,L+1)+(BETA2(L+1)-C2)*B5L(I   ,L))/(BETA2(L
     1+1)-BETA2(L))
      UU2=((C2-BETA2(L))*B5L(I+1,L+1)+(BETA2(L+1)-C2)*B5L(I+1,L))/(BETA2
     1(L+1)-BETA2(L))
      GO TO 83
   82 IF(ALPHA.NE..025)PRINT11
      UU1=((C2-BETA2(L))*B2L(I,L+1)+(BETA2(L+1)-C2)*B2L(I   ,L))/(BETA2(L
     1+1)-BETA2(L))
      UU2=((C2-BETA2(L))*B2L(I+1,L+1)+(BETA2(L+1)-C2)*B2L(I+1,L))/(BETA2
     1(L+1)-BETA2(L))
   83 IF(I.LT.19)UU=((C1-BETA1(I))*UU2+(BETA1(I+1)-C1)*UU1)/(BETA1(I+1)
     1-BETA1(I))
      IF(I.GE.19)UU=UU2

C.....DETERMINE PUI (EQS.3.11)
      CUI=PUI
      CU=(2.*S+1.)*N
      BU=VHU*UU*UU+CU
      IF(PUI.GE..5)BU=VHU*UL*UL+CU
      PUI=(BU+SQRT(BU*BU-   CU*CU))/(2.*N*N)

C.....TEST TO SEE IF ITERATION MUST CONTINUE
      CALL SIGFIG(ISIG,JOK,PLI,PUI,CLI,CUI,KPL,KPU,KCL,KCU)
      FL=ABS(PLI-CLI) $ FU=ABS(PUI-CUI)

C.....IF THE LIMITS AGREE SUFFICIENTLY FOR TWO CONSECUTIVE ITERATIONS,
C.....SET ML=1, ETC.
      IF((FL.LT.FT).OR.(KPL.EQ.KCL))ML=1
      IF((FU.LT.FT).OR.(KPU.EQ.KCU))MU=1
```

```
C.....IF BOTH UPPER AND LOWER LIMITS ARE SUFFICIENTLY CLOSE, DEFINE THE
C.....NUMBER OF THE ITERATION
      IT=M
      IF(ML*MU.NE.1)GO TO 35
   33 IF(PUI.GE.1.)PUI=1.
   73 CONTINUE
      S1LI =CLI $ S2LI=PLI
      S1UI=CUI $ S2UI=PUI
      JT=IT-1
      LT=IT
      RETURN
      END
```

****

```
      SUBROUTINE PEART
C.....THIS SUBROUTINE READS THE PEARSON AND HARTLEY TABLE 42 (VOL.1,
C.....1966) FOR ALPHA=.025 AND .050 AND EXTENDS IT AT EQUAL INCREMENTS
C.....IN BETA1 BY VALUES INTERPOLATED FROM PEARSON AND HARTLEY TABLE 32
C.....(VOL. 2,1972)
      DIMENSION A(100,14),B(400),C( 40,68),D(21,68)
      DIMENSION W5(21),V5(21)
      COMMON/FQ/B2L(21,68),B2U(21,68),B5L(21,68),B5U(21,68)
      DATA(W5=1.6,1.6,1.8,1.8,2.0,2.0,2.2,2.4,2.6,2.8,3.0,3.2,3.6,4.0,
     14.4,4.8,5.2,5.8,6.4,7.0,7.6)
      DATA(V5=9.,9.,9.,9.,9.,9.,9.,9.,9.,9.,9.,9.4,9.8,10.4,11.4,
     111.8,12 6,13.4,14.4)
    7 FORMAT(14(F4.2,1X),2X,F3.1,2(I1))
   10 FORMAT(1X,*IY TOO BIG*)
   11 FORMAT(14(F4.2,1X),1X,F3.1,2(I1))

C.....THIS LOOP READS THE FOUR TABLES
      DO 40 LF=1,4
      IY=0
      IX=0
      Q=-1.

C.....THIS LOOP READS THE CARDS FOR EACH TABLE
      DO 20 I=1,100
      IF(LF.LE.2)READ11, ((A(I,J),J=1,14),B(I),LEVEL,L)
      IF(LF.GT.2)READ7 , ((A(I,J),J=1,14),B(I),LEVEL,L)
      IF(B(I).NE.Q)IY=0
      IF(B(I).NE.Q)IX=IX+1
      IF(B(I).EQ.Q)IY=IY+1
      IF(IY.GT.4)PRINT10
      IF(IY.GT.4)CALL EXIT
      Q=B(I)
      DO 15 J=1,14
      K=14*IY+J
      C(IX,K)=A(I,J)
   15 CONTINUE
      IF(A(I,1).EQ.9.)GO TO 22
   20 CONTINUE
   22 CONTINUE

C.....THIS LOOP ARRANGES THE DATA FROM CARDS ACCORDING TO THE BETA2
C.....VALUES
      DO 50 I=1,21
      J3=INT(5*(V5(I)-1.6+.001))+1
      J1=INT(5*(W5(I)-1.6+.001))
      DO 45 J=1,38
      D(I,J+J1)=C(I,J)
   45 CONTINUE

C.....THIS LOOP ASSIGNS THE UPPERMOST TABULAR VALUE TO THE NON-
C.....CALCULATED SPACE ABOVE
      DO 43 J=1,J1
      D(I,J)=D(I,J1+1)
   43 CONTINUE
```

```
C.....THIS LOOP ASSIGNS THE LOWERMOST TABULAR VALUE TO THE NON-
C.....CALCULATED SPACE BELOW
      DO 44 J=J3,65
      D(I,J)=D(I,J3)
   44 CONTINUE
   50 CONTINUE

C.....THIS LOOP ASSIGNS THE 4 TABLES TO IDENTIFIABLE ARRAYS FOR SUB-
C.....ROUTINE PSA.
      DO 62 I=1,21
      DO 61 J=1,65
      IF(LF.EQ.1)B2L(I,J)=D(I,J)
      IF(LF.EQ.2)B2U(I,J)=D(I,J)
      IF(LF.EQ.3)B5L(I,J)=D(I,J)
      IF(LF.EQ.4)B5U(I,J)=D(I,J)
   61 CONTINUE
   62 CONTINUE
   40 CONTINUE
      RETURN
      END
```

****

```
      SUBROUTINE ANDBUR(PL,PU,PLI,PUI,ISIG,LL)
      COMMON/MUM/PH,QH,RHOH,XLH,N,S,R,T,ALPHA,UA,XLS
      COMMON/SMALL/XLG
      F(A,B,C,N)=(1.-(A/B)**(1./(N-1.)))/(2.-C-(A/B)**(1./(N-1.)))
      G(A,B,C,D,N)=(A*B/(1.+D*((N-2.)*(1.-C)*(1.-C)-2.)-D*D*((N-1.)*
     1(1.-C)*(1.-C)-1.)))**(1./(N-3.))

C.....SLOWER AND SUPPER ARE THE LOWER AND UPPER CONFIDENCE LIMITS FROM
C.....THE PEARSON AND HARTLEY TABLE FOR THE EXPECTATION OF A POISSON
C.....VARIABLE (1966, TABLE 40)

      CALL PH40(SLOWER,SUPPER)

C.....PLI AND PUI ARE THE ANDERSON-BURSTEIN LIMITS FOR INDEPENDENT
C.....TRIALS.
      PLI=SLOWER/(N-(S-1.-SLOWER)/2.)
      PUI=SUPPER/(N+(SUPPER-S)/2.)
      SS=SQRT((1.+RHOH)/(1.-RHOH))

C.....PL AND PU ARE THE MODIFIED ANDERSON-BURSTEIN LIMITS FOR DEPENDENT
C.....TRIALS
C.....SEE EQS. 6.1 AND 6.2
      PU=PH+(PUI-PH)*SS
      PL=PH-(PH-PLI)*SS
      IF((PL.LE.0.))PU=(PUI-PLI)*SS
      IF(PL.LT.0.)PL=0.
      IF(PU.GT.1.)PU=1.
   30 CONTINUE

C.....IF S=0,1, THE EQUATIONS (7.3, 7.4) FOR THE EXACT SOLUTION WILL BE
C.....IMPLEMENTED BY THE FOLLOWING STATEMENTS. OTHERWISE THE VALUES
C.....OF PL AND PU, JUST DETERMINED, WILL BE RETAINED.
C.....LL=NUMBER OF DECIMAL PLACES FOR AGREEMENT ON CONSECUTIVE
C.....ITERATIONS WHEN S=0,1
      FT=1.*10**(-LL)
      ML=MU=0
   15 IF(S.EQ.0.)20,21
   20 PL=CL=0.
      CU=PU
      QU=1.-PU
      PU=F(ALPHA,QU,XLG,N)
      GO TO 27
   21 IF(S.EQ.1.)22,23
   22 PL=CL=0.
      CU=PU
      QU=1.-PU
      XU=PU
      PU=(1.-G(ALPHA,QU,XLG,XU,N))/(2.-XLH-G(ALPHA,QU,XLG,XU,N))
   27 CONTINUE
      CALL SIGFIG(ISIG,JOK,PL,PU,CL,CU,KPL,KPU,KCL,KCU)
      FL=ABS(PL-CL) $ FU=ABS(PU-CU)

C.....IF THE LIMITS AGREE SUFFICIENTLY FOR TWO CONSECUTIVE IT-
C.....ERATIONS, SET ML=1, ETC
      IF((FL.LT.FT).OR.(KPL.EQ.KCL))ML=1
      IF((FU.LT.FT).OR.(KPU.EQ.KCU))MU=1
      IF(ML*MU.NE.1)GO TO 15
   23 CONTINUE
      RETURN
      END

*************************************************************************
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```
      SUBROUTINE PH40(SLOWER,SUPPER)

C.....THIS SUBROUTINE DETERMINES THE 80, 90, 95, AND 99 PERCENT
C.....CONFIDENCE LIMITS FOR ERRORS USING THE PEARSON AND HARTLEY TABLES
C.....OF CONFIDENCE LIMITS FOR THE EXPECTATION OF A POISSON VARIABLE
C.....(1966, TABLE 40). THE TABLE IS APPLICABLE TO ERRORS BETWEEN 0 AND
C.....100 FOR 80 AND 90 PERCENT LIMITS, 0 AND 50 FOR 95 AND 99 PERCENT
C.....LIMITS. THE TABULAR VALUES ARE ALOWER AND AUPPER.
C.....THE CONFIDENCE LIMITS ARE SLOWER AND SUPPER. INTERPOLATION IS REQ-
C.....UIRED FOR MORE THAN 30 ERRORS SINCE ONLY EVERY FIFTH ERROR IS
C.....LISTED AFTER 30 ERRORS.
      DIMENSION ALOWER(45), AUPPER (45),BLOWER(45),BUPPER(45),FLOWER(45)
     1,FUPPER(45),GLOWER(45),GUPPER(45)
      COMMON/MJM/PH,QH,RHOH,XLH,N,S,R,T,ALPHA,UA,XLS

C.....99 PERCENT LIMITS
      DATA(FLOWER=0.,.005,    .103,.338,.672,1.08,1.54,2.04,2.57,3.13,
     13.72,4.32,4.94,5.58,6.23,6.89,7.57,8.25,8.94,9.64,10.35,11.07,
     111.79,12.52,13.25,14.00,14.74,15.49,16.24,17.00,17.77,21.64,
     125.59,29.60,33.66)
      DATA(FUPPER=5.30,7.43,9.27,10.98,12.59,14.15,15.66,17.13,18.58,
     120.00,21.40,22.78,24.14,25.50,26.84,28.16,29.48,30.79,32.09,
     133.38,34.67,35.95,37.22,38.48,39.74,41.00,42.25,43.50,44.74,
     145.98,47.21,53.32,59.36,65.34,71.27)

C.....95 PERCENT LIMITS
      DATA(BLOWER=0.,.0253,.242,.619,1.09,1.62,2.20,2.81,3.45,4.12,4.80
     1,5.49,6.20,6.92,7.65,8.40,9.15,9.90,10.67,11.44,12.22,13.00,13.79,
     114.58,15.38,16.18,16.98,17.79,18.61,19.42,20.24,24.38,28.58,
     132.82,37.11)
      DATA(BUPPER=3.69,5.57,7.22,8.77,10.24,11.67,13.06,14.42,15.76,
     117.08,18.39,19.68,20.96,22.23,23.49,24.74,25.98,27.22, 28.45,29.67
     1,30.89,32.10,33.31,34.51,35.71,36.90,38.10,39.28,40.47,41.65,42.
     183,48.68,54.47,60.21,65.92)

C.....90 PERCENT LIMITS
      DATA (ALOWER = 0., .0513, .355, .818, 1.37, 1.97, 2.61, 3.29, 3.98
     1, 4.70, 5.43, 6.17, 6.92, 7.69, 8.46, 9.25, 10.04, 10.83, 11.63, 1
     22.44, 13.25, 14.07, 14.89, 15.72, 16.55, 17.38, 18.22, 19.06, 19.9
     30, 20.75, 21.59, 25.87, 30.20, 34.56, 38.96, 43.40, 47.85, 52.33,
     456.83, 61.35, 65.88, 70.42, 74.98, 79.56, 84.14)
      DATA (AUPPER = 3.00, 4.74, 6.30, 7.75, 9.15, 10.51, 11.84, 13.15,
     114.43, 15.71, 16.96, 18.21, 19.44, 20.67, 21.89, 23.10, 24.30, 25.
     250, 26.69, 27.88, 29.06, 30.24, 31.42, 32.59, 33.75, 34.92, 36.08,
     3 37.23, 38.39, 39.54, 40.69, 46.40, 52.07, 57.69, 63.29, 68.85, 74
     44.39, 79.91, 85.40, 90.89, 96.35, 101.80, 107.24, 112.67, 118.00)

C.....80 PERCENT LIMITS
      DATA(GLOWER=0.,.105,.53,1.10,1.74,2.43,3.15,3.9,4.7,5.4,6.2,7.0,
     17.8,8.6,9.5,10.3,11.1,12.0,12.8,13.7,14.5,15.4,16.2,17.1,18.0,
     118.8,19.7,20.6,21.5,22.3,23.2,27.7,32.1,36.6,41.2,46.,50.,55.,60.,
     164.,69.,73.,78.,83.,87.)
      DATA(GUPPER=2.30,3.9,5.3,6.7,8.0,9.3,10.5,11.8,13.0,14.2,15.4,
     116.6,17.8,19.0,20.1,21.3,22.5,23.6,24.8,25.9,27.0,28.2,29.3,30.5,
     131.6,33.,34.,35.,36.,37.,38.,44.,49.,55.,60.,66.,71.,77.,82.,87.,
```

```
     193.,98.,103.,109.,114.)
      IERROR=INT(S+1.E-8)
      XE=S
      JERROR = IERROR + 1

C.....IF INTERPOLATION IS REQUIRED, GO TO 105.
      IF (IERROR .LE. 30)100, 105
  100 IF(ALPHA.EQ..005)SLOWER=FLOWER(JERROR)
      IF(ALPHA.EQ..005)SUPPER=FUPPER(JERROR)
      IF(ALPHA.EQ..025)SLOWER=BLOWER(JERROR)
      IF(ALPHA.EQ..025)SUPPER=BUPPER(JERROR)
      IF(ALPHA.EQ..05) SLOWER=ALOWER(JERROR)
      IF(ALPHA.EQ..05) SUPPER=AUPPER(JERROR)
      IF(ALPHA.EQ..10)SLOWER=GLOWER(JERROR)
      IF(ALPHA.EQ..10)SUPPER=GUPPER(JERROR)
      GO TO 110
  105 I1 = (IERROR / 5) * 5
      I2 = ((IERROR + 5) / 5) * 5
      J1 = I1 - ((I1 - 30) * 8) / 10 + 1
      J2 = J1 + 1
      IF(ALPHA.EQ..005)SLOWER=FLOWER(J2)*(XE-I1)/5.+FLOWER(J1)*(I2-XE)/
     15.
      IF(ALPHA.EQ..005)SUPPER=FUPPER(J2)*(XE-I1)/5.+FUPPER(J1)*(I2-XE)/
     15.
      IF(ALPHA.EQ..025)SLOWER=BLOWER(J2)*(XE-I1)/5.+BLOWER(J1)*(I2-XE)/
     15.
      IF(ALPHA.EQ..025)SUPPER=BUPPER(J2)*(XE-I1)/5.+BUPPER(J1)*(I2-XE)/
     15.
      IF(ALPHA.EQ..050)SLOWER=ALOWER(J2)*(XE-I1)/5.+ALOWER(J1)*(I2-XE)/
     15.
      IF(ALPHA.EQ..050)SUPPER=AUPPER(J2)*(XE-I1)/5.+AUPPER(J1)*(I2-XE)/
     15.
      IF(ALPHA.EQ..100)SLOWER=GLOWER(J2)*(XE-I1)/5.+GLOWER(J1)*(I2-XE)/
     15.
      IF(ALPHA.EQ..100)SUPPER=GUPPER(J2)*(XE-I1)/5.+GUPPER(J1)*(I2-XE)/
     15.
  110 CONTINUE
      RETURN
      END
```

****

PARAMETER VALUES.....,N= 20000 S= 38 R= 13 T= 0 ALPHA=.050
ACCURACY VALUES.....MINSIG=3 MINDEC=4 MAXIT=20

TANGENTS OF 90 PERCENT CONFIDENCE REGION

	HORIZONTAL TANGENT (LIMITS FOR P)		VERTICAL TANGENT (LIMITS FOR LAMBDA)	
	P	LAMBDA	P	LAMBDA
UPPER	3.67381E-03	4.54605E-01	2.34393E-03	5.29429E-01
LOWER	1.13989E-03	2.94945E-01	1.47628E-03	1.84585E-01

90 PERCENT CONFIDENCE INTERVAL FOR LAMBDA = (2.35021E-01, 4.81484E-01)

90 PERCENT CONFIDENCE INTERVAL FOR P

ANDERSON-BURSTEIN = (1.21599E-03, 2.74639E-03)

NORMAL = (1.27837E-03, 2.80949E-03)

EDGEWORTH 2-TERM ITERATION NO. 1= (1.24223E-03, 2.76358E-03)
EDGEWORTH 2-TERM ITERATION NO. 2= (1.24123E-03, 2.76451E-03)

EDGEWORTH 4-TERM ITERATION NO. 1= (1.24939E-03, 2.76886E-03)
EDGEWORTH 4-TERM ITERATION NO. 2= (1.24800E-03, 2.76992E-03)

PEARSON SYSTEM ITERATION NO. 2 = (1.24787E-03, 2.75997E-03)
PEARSON SYSTEM ITERATION NO. 3 = (1.24786E-03, 2.75997E-03)

P HAT= 1.90000E-03
LAMBDA TILDE= 3.48160E-01

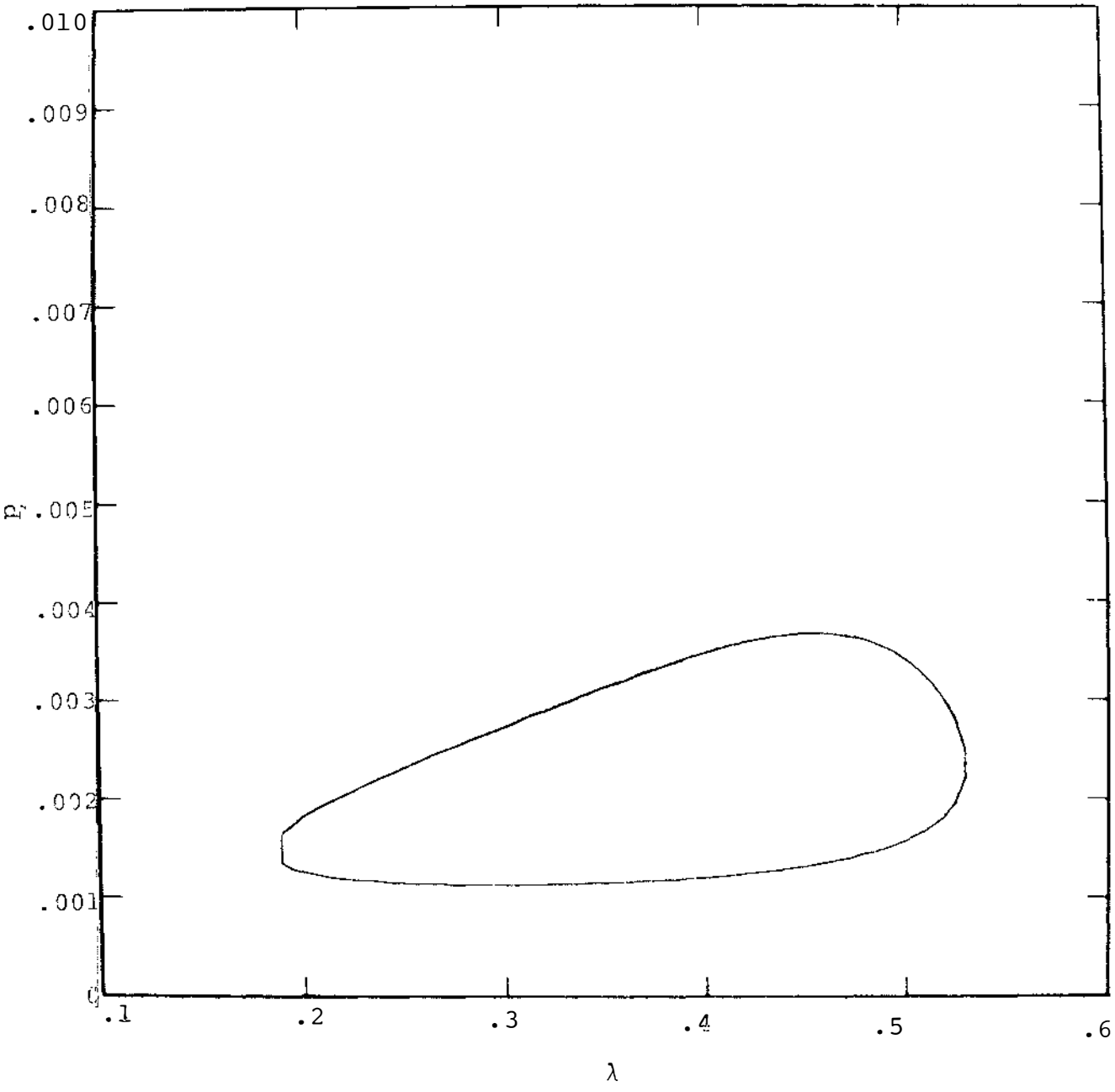

Figure B-1. Microfilm plot of 90% confidence region for (λ,p) from first 20,000 Cox-Lewis telephone data and normal approximation (38 errors, $\hat{p}=.0019$, $\tilde{\lambda}=.3482$).

FORM OT-29
(3-73)

U.S. DEPARTMENT OF COMMERCE
OFFICE OF TELECOMMUNICATIONS

BIBLIOGRAPHIC DATA SHEET

	1. PUBLICATION OR REPORT NO. OTR 77-118	2. Gov't Accession No.
3. Recipient's Accession No.		

4. TITLE AND SUBTITLE
CONFIDENCE LIMITS FOR DIGITAL ERROR RATES FROM DEPENDENT TRANSMISSIONS

5. Publication Date
March, 1977

6. Performing Organization Code
OT/ITS

7. AUTHOR(S)
Edwin L. Crow and Martin J. Miles

9. Project/Task/Work Unit No.

8. PERFORMING ORGANIZATION NAME AND ADDRESS
U.S. Department of Commerce
Office of Telecommunications
Institute for Telecommunication Sciences
325 Broadway, Boulder, CO 80302

10. Contract/Grant No.

11. Sponsoring Organization Name and Address
U.S. Department of Commerce
Office of Telecommunications
1325 G Street, NW
Washington, DC 20005

12. Type of Report and Period Covered

13.

14. SUPPLEMENTARY NOTES

15. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

Approximate confidence limits for error rates (probabilities of an error) of digital communication systems are derived under the assumptions that the sample is large, the error rate is constant and the dependence between transmissions is governed by a stationary first-order Markov chain model. Five different approximations are given: normal distribution, Edgeworth two-term and four-term, Pearson system, and modified Poisson-Anderson-Burstein. Results by Gabriel and by Klotz are used. The limits are compared with Ladd's small-sample limits and illustrated on telephone data. Methods for testing the validity of the model are described. A computer program for applying the limits is presented, and an extensive Monte Carlo simulation to test the accuracy of the limits is summarized. Methods for designing the experiment so as to achieve a specified precision are given.

16. Key Words *(Alphabetical order, separated by semicolons)*

Bernoulli trials, chi-squared test, confidence limits, confidence region, region, design of experiments, digital communication systems, Edgeworth series, Freeman-Tukey deviates, likelihood radio test, Markov chain, Monte Carlo simulation, nonstationarity, Pearson system

17. AVAILABILITY STATEMENT

[X] UNLIMITED.

[ ] FOR OFFICIAL DISTRIBUTION.

18. Security Class *(This report)* Unclassified	20. Number of pages 156
19. Security Class *(This page)* unclassified	21. Price: