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Statistical Considerations for Noise and Interference Measurements

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STATISTICAL CONSIDERATIONS FOR NOISE AND INTERFERENCE MEASUREMENTS

Roger Dalke*

The study of noise and interference in the radio environment is essential to the development of efficacious communications systems. Many of the characteristics of radio noise and interference of interest to radio system designers can be expressed in terms of first- and second-order statistics. These statistics are necessarily calculated using measured data. In this report, we discuss uncertainties that arise when measured data sets are used to calculate statistics of radio noise and interference.

Key words: amplitude probability distribution; average power; measurement uncertainties; power spectrum; radio interference; radio noise

1. INTRODUCTION

The performance of wireless communications systems is adversely affected by a variety of environmental processes which from the point of view of a victim receiver are time-varying stochastic processes. These processes can be either additive or multiplicative. Examples of additive processes are natural and man-made noise and interference. Multiplicative processes involve radio channel effects that result in signal fading such as multipath. In what follows, we will simply use the term noise to describe these random processes.

Since communications systems need to resist noise, it is important that the salient characteristics of these processes be well understood. For many applications of interest to communications engineers, these characteristics can be expressed as properties of their first- and second-order statistics. Such statistics are necessarily determined via measurements. Measurements, however, only provide finite data sets which introduce uncertainties in calculated statistics.

The intent of this report is to discuss some of the statistical uncertainties that arise when measured data are used to characterize radio noise and interference. In particular, we discuss uncertainties for estimates of mean power, statistical distribution functions (viz. the *amplitude probability distribution*), covariance functions and power spectral density functions. It should be noted that while the concepts described here are not new, they are presented in a format that is intended to provide a useful reference for communications engineers performing noise and interference measurements. For a more detailed discussion of the topics

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described in this report, the interested reader is directed to the list of publications in the reference section.

Section 2 of this report gives a brief overview of relevant concepts from the mathematical theory of stochastic processes. This section also gives mathematical expressions for functions and definitions used throughout the report. Estimators and related uncertainties for commonly measured first order statistics are described in Sections 3 and 4. Section 5 describes the statistical properties of commonly used estimators for the covariance and power spectral density functions.

2. PRELIMINARY REMARKS ABOUT STOCHASTIC PROCESSES

The study of noise in the radio environment is essential to the development of efficacious communications systems. From the standpoint of a victim receiver, noise processes are generally random functions in time and space. The development of appropriate noise measurement techniques and the proper interpretation of results is greatly enhanced by an understanding of the mathematical theory of stochastic processes. The synergism of theory and measurements allows engineers to develop a keen understanding of the probabilistic structure of important noise processes and provides a basis for analyzing the performance of radio systems. In this section we provide a brief discussion on pertinent aspects of the mathematical description of stochastic processes.

Formally a stochastic process is defined in terms of a probability space $(\Omega, \mathfrak{S}, \mathcal{P})$ and a parameter set $\mathfrak{T}$ representing, for example, time variation (see [1] for a complete description of this and what follows). Ω is the space of elementary events ω (e.g., possible outcomes of an experiment), $\mathfrak{S}$ is a σ-field of sets in Ω, and $\mathcal{P}$ is a probability measure. A stochastic process is a finite function $\xi(t, \omega)$ which, for every fixed $t \in \mathfrak{T}$, is a measurable function of $\omega \in \Omega$.

With regard to the use of the independent variable ω, we will adopt practices commonly found in the literature. Specifically, the independent variable ω will be suppressed unless it is required for calculations or it provides clarification. At times it will be useful to denote the realization of a random variable or process based on a particular elementary event. For example, $\xi(t, \omega_1)$ denotes the realization (or sample function) resulting from the event $\{\omega = \omega_1\}$. Throughout this report $\xi(t)$ is used to denote a random process. The specific type or class of random process represented by the function $\xi(t)$ will be defined as required in each section or subsection.

For a fixed $t = t_1$ there is a random variable $\xi(t_1) = \xi(t_1, \omega)$ with a definite probability distribution denoted by

$$F(x; t_1) = \mathcal{P}\{\xi(t_1) \leq x\}$$

that is, $F(x; t_1)$ is the probability of the event that the random experiment yields a value of $\xi(t_1)$ which does not exceed the real number x. For an arbitrary finite set of t-values, $t_1, \ldots, t_n$, the random variables $\xi(t_1), \ldots, \xi(t_n)$ have the joint distribution function

$$F(x_1, \ldots, x_n; t_1, \ldots, t_n) = \mathcal{P}\{\xi(t_1) \leq x_1, \ldots, \xi(t_n) \leq x_n\}.$$

All joint probability distributions for all $n = 1, 2, \ldots$ and all possible values of t_i constitutes the family of finite-dimensional distributions needed to completely characterize the process $\xi(t, \omega)$. It is clear that completely characterizing many of the random noise processes of interest in communications engineering would be quite difficult if not impossible. Evidently a random process is described in terms of a denumerable number of random variables. In this context, the statistical properties of one random variable (e.g., $F(x_1; t_1)$) are usually referred to as first-order statistics and the joint statistics of 2 random variables (e.g., $F(x_1, x_2; t_1, t_2)$) are called second-order statistics.

Fortunately, it is often the case that first- and second-order statistics provide sufficient information for predicting how noise affects the performance of communication systems. In addition to the mean,

$$\mu(t) = \mathcal{E}\{\xi(t)\}$$

where $\mathcal{E}$ denotes expected value, the usual first-order statistic of interest is the *amplitude probability distribution* (APD) which is the compliment of the distribution function, i.e., $1 - F(x;t)$, for the process amplitudes.

Second-order statistics of interest are the *correlation function*

$$r(t_1, t_2) = \mathcal{E}\{\xi(t_1)\xi^*(t_2)\},$$

the *covariance function*

$$\text{cov}(\xi(t_1), \xi(t_2)) = \gamma(t_1, t_2) = \mathcal{E}\{\xi(t_1)\xi^*(t_2)\} - \mathcal{E}\{\xi(t_1)\}\mathcal{E}\{\xi^*(t_2)\}, \tag{1}$$

the *variance*

$$\text{var}(\xi(t)) = \text{cov}(\xi(t), \xi(t))$$

and the Fourier transform of the covariance function

$$\Gamma(f_1, f_2) = \mathscr{F}\{\gamma(t_1, t_2)\}.$$

The superscript * used above denotes complex conjugate.

If the process of interest is discrete, the time parameter is usually denoted by an integer subscript (e.g., $\xi_k(\omega)$). The extension of the previous equations to discrete processes is obvious, for example the discrete covariance is

$$\gamma_{ij} = \mathcal{E}\left\{\xi_i \xi_j^*\right\} - \mathcal{E}\{\xi_i\}\mathcal{E}\left\{\xi_j^*\right\}. \tag{2}$$

At this point it is useful to introduce some standard terminology used in the text. Let M denote the average of N random samples of the random variable ξ where $\mathcal{E}\{\xi\} = \mu$ (i.e., M is an estimator for the mean). If $\mathcal{E}\{M\} = \mu$ the estimator is called *unbiased*. If $\mathcal{E}\{M\} \neq \mu$, but we have $\mathcal{E}\{M\} \to \mu$ as $N \to \infty$ then the estimator is said to be *asymptotically unbiased*. It is also important to describe convergence properties of the estimator. *Mean-square convergence* means that $\text{var}(M) = \mathcal{E}\{(M-\mu)^2\} \to 0$ as $N \to \infty$. If in addition we have $\mathcal{P}\{|M - \mu| < \epsilon\} = 1$ for any $\epsilon > 0$ then we will say that $M \to \mu$ with probability one or in probability. An estimate that converges in probability is also called a *consistent* estimate. Note that convergence in probability is a stronger condition than mean-square convergence (see for example the random sampling variance calculation in Section 3.2.4). We will sometimes use the standard symbol O when describing the order of magnitude of a function when the independent variable tends to some limit (usually infinity). For example $\text{var}(M) = O(N^{-1})$ means that $N\text{var}(M)$ remains bounded as $N \to \infty$.

2.1 Stationarity and Ergodicity

The family of processes that are invariant under a translation in time (or space) are called *stationary processes*. This property greatly simplifies measurement procedures. Different classes of stationarity are obtained by imposing more or less stringent conditions of invariance. A process is *strictly stationary* when the whole family of finite-dimensional distributions is invariant under a translation in the parameter of interest (e.g., time). Communications engineers are often concerned with the broader class of processes that have time invariant first- and second-order statistics. This class is usually referred to as *wide-sense stationary*. In the sequel, these will simply be called stationary processes. Thus, for stationary processes we have the following relations

$$\mu(t) = \mu \quad F(x;t) = F(x) \quad r(t_1,t_2) = r(t_1 - t_2) \quad \gamma(t_1,t_2) = \gamma(t_1 - t_2) \quad \Gamma(f) = \mathscr{F}\{\gamma(t_1 - t_2)\}.$$

Since the variance of a stationary process plays an important role in characterizing uncertainties we will at times simply call it σ^2, i.e., $\sigma^2 = \gamma(0)$. Note that in the case of the covariance and correlation function, we have used the same symbol to represent both a function of two variables and a single variable which is the difference of the two independent variables. We will frequently use such simplified notation when the relationship between the two functions is obvious as with stationary processes.

While, from the standpoint of performing experiments, the attribute of stationarity is highly desirable, many of the noise processes encountered in radio communications are nonstationary. For example, radio noise originating from human activity can vary significantly within the hour as well as hour-to-hour. However, it is usually possible to adequately characterize these processes using measurements over time scales during which the process is approximately stationary. For longer time scales, the first- and higher-order statistics are usually treated as random variables.

In principle, the statistical properties of a noise process can be calculated from an ensemble of sample functions obtained from a sufficiently large number of random events. Implementing the experiments required to construct such an ensemble is usually not practical and may not even be possible. An obvious and more practical procedure is to try and obtain the desired statistics from a sample function $\xi(t) = \xi(t,\omega_1)$ corresponding to a single random experiment. Such a procedure is valid if each sample function is representative, when viewed over sufficiently long times, of the statistical properties of the entire ensemble—and conversely. Random processes with this property are known as *ergodic* processes. In cases where it is only possible to measure a single sample function, and the process is known to be stochastic, the usual practice is to assume the process is ergodic and construct the desired statistics from the measured sample function.

Formally, for a strictly stationary discrete time process $\xi_k(\omega)$ with a bounded first moment ($\mathcal{E}\{|\xi_k|\} < \infty$), a point in the basic probability space $\omega = (\omega_0, \omega_1, \omega_2, \dots)$ can be thought of as a possible time history of an event, i.e., $\xi_k(\omega) = \omega_k$. If $\mathcal{T}$ is a time shift transformation, $(\mathcal{T}\omega)_k = \omega_{k+1}$, and if for every invariant set $A \in \Omega$ (i.e., $\mathcal{T}^{-1}A = A$) $\mathcal{P}\{A\} = 0$ or 1, the

ergodic theorem [2] states that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \xi_0(\mathcal{T}^k\omega) = \mathcal{E}\{\xi_0\}$$

with probability one.

More generally, this includes time averages of bounded (in the first moment) integrable functions f of the process, i.e.,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(\mathcal{T}^k\omega) = \mathcal{E}\{f\}.$$

For example, if the first-order distribution function $F(x)$ is estimated by a function F_x which gives the fraction of time that the observed sample function is less than a particular level x, then ergodicity implies that $F_x \to F(x)$ as $N \to \infty$.

2.1.1 A Strictly Stationary and Ergodic Process

Consider the process $\xi_0, \xi_1, \xi_2 \ldots$ where the $\{\xi_k\}$ are independent and identically distributed (iid) random variables with $\mathcal{E}\{|\xi_k|\} < \infty$ and form the time average

$$M = \frac{1}{N} \sum_{k=0}^{N-1} \xi_k.$$

By the *strong law of large numbers* $|M - \mathcal{E}\{\xi_k\}| \to 0$ with probability one.

2.1.2 A Strictly Stationary Process that is Not Ergodic

Let ω be uniformly distributed over the unit interval ($\omega \in [0, 1)$) and define the random variable X as a real function that can be represented as the Fourier series

$$X(\omega) = \sum_{n=-\infty}^{+\infty} c_n e^{i2\pi n\omega}$$

where $i = \sqrt{-1}$. For any $\lambda \in [0, 1)$ and transformation $\mathcal{T}\omega = \omega + \lambda \pmod 1$ obtain the random process

$$\xi_k(\omega) = X(\mathcal{T}^k\omega) \quad k = 0, 1, 2, \ldots.$$

Since T is a measure preserving transformation ($\mathcal{P}\{\mathcal{T}A\} = \mathcal{P}\{A\}$ for any set $A \in \Omega$), the process is stationary and $\mathcal{E}\{\xi_k\} = c_0$. The time average is

$$M = \sum_{n\lambda\in\mathbb{Z}} c_n e^{i2\pi\omega n} + \frac{1}{N} \sum_{n\lambda\notin\mathbb{Z}} c_n e^{i2\pi\omega n} \left(\frac{e^{i2\pi Nn\lambda} - 1}{e^{i2\pi n\lambda} - 1} \right),$$

where $\mathbb{Z}$ is the set of integers. In the limit $N \to \infty$ we have

$$M \to \sum_{n\lambda \in \mathbb{Z}} c_n e^{i2\pi\omega n}.$$

If λ is a rational number, the process is periodic and there are non-empty invariant sets that do not have probabilities of 0 or 1. Hence, the transformation is not ergodic. Clearly the time average M does not in general converge to c_0.

If, on the other hand, λ is irrational, the process is not periodic and it can be shown that the invariant sets have probability 1 or 0. The transformation is ergodic and $M \to c_0$ with probability one [2].

3. ESTIMATION OF THE MEAN

Perhaps the first order statistic that is most widely used to characterize radio noise is the mean power. In this section, we will examine the statistical properties of the commonly used power estimator. Expressions for the mean and variance of the estimator will be obtained. We will see that the variance depends on correlation properties of the data and hence data sampling strategies. By way of an example, these results will be applied to narrowband Gaussian radio noise.

When the distribution of the estimator is known or can be reasonably approximated, a more precise statistical description of uncertainties can be obtained. An example of such a procedure, the method of confidence intervals, is also given in this section.

Finally, we examine the mean and variance of the estimator when the underlying process is cyclostationary. This class of nonstationary processes often occurs in the analysis of communications systems. This is followed by an application of the results to a pulse modulated process.

3.1 Stationary Processes

In this subsection, we describe uncertainties associated with estimating the mean of a stationary process. Let $\xi(t,\omega)$ be such a process with mean $\mu = \mathcal{E}\{\xi(t)\}$. The usual procedure is to calculate the time average of a measured sample function $\xi(t,\omega_1)$ based on a particular experiment ω_1. If we obtain N discrete samples equally spaced Δt time units apart and denote $\xi_k(\omega_1) = \xi(k\Delta t, \omega_1)$, the estimate of the mean is

$$M(\omega_1) = \frac{1}{N}\sum_{k=0}^{N-1} \xi_k(\omega_1). \tag{3}$$

Note that the estimator M is a function on the probability space consisting of elementary events $\omega \in \Omega$, i.e., it is a random variable. Hence, the characteristics of the estimate are properly described in terms of its statistics.

The mean of the estimate is

$$\mathcal{E}\{M\} = \frac{1}{N}\sum_{k=0}^{N-1} \mathcal{E}\{\xi_k\} = \mu\,, \tag{4}$$

hence M is an unbiased estimate. The variance is

$$\mathrm{var}(M) = \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} \mathcal{E}\{\xi_j \xi_k^*\} - |\mu|^2 = \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} \gamma_{jk}\,, \tag{5}$$

where γ_{jk} is given by Equation 2.

For a real stationary process $\gamma_{jk} = \gamma_n$, where $n = k - j$, so that Equation 5 simplifies to

$$\mathrm{var}(M) = \frac{\sigma^2}{N} + \frac{2}{N^2} \sum_{n=1}^{N} (N - n)\gamma_n$$

where $\sigma^2 = \gamma_0$.

If the process is stationary and

$$\frac{1}{N} \sum_{k=0}^{N-1} \gamma_k \to 0$$

the estimator is mean-square convergent. This is sometimes referred to as mean-square ergodicity. If in addition $\gamma(t) = O(|t|^{-\beta})$ for some $\beta > 0$ as $|t| \to \infty$, then $M \to \mu$ with probability one (i.e., the estimate is consistent) [1]. It should also be noted that if $\gamma_n = 0$ for $n \neq 0$, the variance of the estimator is simply

$$\mathrm{var}(M) = \frac{\sigma^2}{N}. \tag{6}$$

3.1.1 Sample Correlation Effects

As we have seen, when the samples are uncorrelated, the variance of the estimator decreases in proportion to the number of data samples. In this subsection we will compare this result with what would be obtained using a fixed length record of densely sampled data. In fact, we will assume that the number of samples is so large that the summation in Equation 3 can be replaced with an integral.

To obtain uncorrelated samples we need to know or otherwise estimate the minimum sample spacing τ_0 that yields (at least approximately) uncorrelated samples. This sample spacing is often referred to as *correlation time* and we assume that $\gamma(\tau) \approx 0$ when $\tau \geq \tau_0$. A practical definition of correlation time is

$$\frac{\tau_0}{2} = \frac{1}{\gamma(0)} \mathrm{Re} \int_0^\infty \gamma(\tau) \mathrm{d}\,\tau. \tag{7}$$

As might be anticipated, there is a relationship between correlation time and the bandwidth of the power spectrum. If the equivalent bandwidth of the power spectrum $\Gamma(f)$ is defined as

$$B_{eq} = \frac{1}{\Gamma(0)} \int_{-\infty}^{\infty} \Gamma(f) \mathrm{d}f$$

then

$$2\tau_0 = \frac{1}{B_{eq}}.$$

In the case of a triangular function, $2\tau_0$ is exactly equal to the width of the normalized covariance function. More generally, the quantity $2\tau_0$ can be considered as a measure of the width of the covariance function.

If the time increment is greater than the correlation time, we can simply calculate the variance using Equation 6. Note that evaluating the variance requires a knowledge of σ^2 which is often not directly available. When N is large, a common practice is to replace σ^2 with the unbiased estimate of the variance calculated directly from the measured data

$$S^2 = \frac{1}{N-1}\sum_{k=0}^{N-1}(\xi_k - M)^2 \tag{8}$$

which converges to σ^2 in probability.

If a real stationary and ergodic noise process is densely sampled over a time period T, the variance of the estimate can be calculated via integration

$$\mathrm{var}(M) = \frac{1}{T^2}\int_0^T\int_0^T \gamma(s-t)\,\mathrm{d}t\,\mathrm{d}s = \frac{1}{T}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)\gamma(\tau)\mathrm{d}\,\tau. \tag{9}$$

If τ_0 is finite and $T >> \tau_0$ we have

$$\mathrm{var}(M) \approx \frac{2}{T}\int_0^\infty \gamma(\tau)\mathrm{d}\,\tau = \frac{\gamma(0)}{T/\tau_0}. \tag{10}$$

But T/τ_0 is approximately equal to the *maximum* number of *uncorrelated* samples contained in the sample time period. Hence, the variance for dense sampling is not much different than what would be obtained for sparse sampling using a sample spacing of τ_0.

3.1.2 An Example – Narrowband Noise

In this subsection we will use the results obtained so far to calculate the variance of mean power estimators assuming densely and sparsely sampled narrowband Gaussian noise.

Let $n(t)$ be a stationary narrowband Gaussian noise process with $\mathcal{E}\{n(t)\} = 0$, center frequency f_c, and baseband representation $\hat{n}(t)$, i.e.,

$$n(t) = \mathrm{Re}\,\hat{n}(t)e^{-i2\pi f_c t}.$$

Let $\xi(t) = |\hat{n}(t)|^2$ and assume that we wish to estimate the mean power $\mu = \mathcal{E}\{\xi(t)\}$ using Equation 3. We know from Equation 4 that the estimate is unbiased. The variance of the estimate depends on the covariance of $\xi(t)$, so we will first obtain an expression for the covariance in terms of the baseband noise process. From Equation 1 we immediately obtain

$$\gamma(s-t) = \mathcal{E}\{\hat{n}(t)\hat{n}^*(t)\hat{n}(s)\hat{n}^*(s)\} - \mu^2$$

This result can be simplified using the following identity for stationary zero mean Gaussian processes [3]

$$\mathcal{E}\left\{\hat{n}(t)\hat{n}^*(s)\hat{n}^*(t)\hat{n}(s)\right\} = \mathcal{E}\left\{\hat{n}(t)\hat{n}^*(s)\right\}\mathcal{E}\left\{\hat{n}^*(t)\hat{n}(s)\right\} + \mathcal{E}\left\{\hat{n}(t)\hat{n}(s)\right\}\mathcal{E}\left\{\hat{n}^*(t)\hat{n}^*(s)\right\} + \mu^2$$

to obtain

$$\gamma(\tau) = |\mathcal{E}\left\{\hat{n}(t)\hat{n}^*(s)\right\}|^2 + |\mathcal{E}\left\{\hat{n}(t)\hat{n}(s)\right\}|^2$$

where $\tau = s - t$.

Often the noise equivalent bandwidth is known (at least approximately) so it is useful to formulate the variance of the estimator in terms of this parameter. We begin by noting that narrowband Gaussian noise can be represented as bandpass filtered white noise $z(t)$ with spectral density $k\mathscr{T}$ where k is Boltzmann's constant ($1.3807 \times 10^{-23}\mathrm{JK}^{-1}$) and $\mathscr{T}$ is the temperature in Kelvin. If $\hat{h}$ is the baseband representation of the bandpass filter, we have

$$\hat{n}(t) = \int_{-\infty}^{\infty} z(s)e^{i2\pi f_c s}\hat{h}(t-s)\mathrm{d}s.$$

Since $\mathcal{E}\left\{z(t)z(s)\right\} = k\mathscr{T}\delta(s-t)$ we find

$$\mathcal{E}\left\{\hat{n}(t)\hat{n}^*(s)\right\} = k\mathscr{T}\int_{-\infty}^{\infty}\hat{h}(\tau+s)\hat{h}^*(s)\mathrm{d}s$$

and

$$\mathcal{E}\left\{\hat{n}(t)\hat{n}(s)\right\} = k\mathscr{T}e^{i4\pi f_c t}\int_{-\infty}^{\infty} e^{-i4\pi f_c s}\hat{h}(\tau+s)\hat{h}(s)\mathrm{d}s.$$

Using the notation $\hat{H}(f) = \mathscr{F}\{\hat{h}(t)\}$, the integral given above can readily be evaluated as

$$\int_{-\infty}^{\infty}\hat{H}(2f_c - g)\hat{H}(g)e^{-i2\pi g\tau}\mathrm{d}g \equiv 0$$

and hence $\mathcal{E}\left\{\hat{n}(t)\hat{n}(s)\right\} \equiv 0$ so that

$$\gamma(\tau) = \left|k\mathscr{T}\int_{-\infty}^{\infty}\hat{h}(\tau+s)\hat{h}^*(s)\mathrm{d}s\right|^2.$$

Usually, the filter is specified so that the maximum gain is 1 (i.e., $\max|\hat{H}(f)|^2 = 1$. The mean and variance of $\xi(t)$ can then be expressed in terms of the filter's equivalent bandwidth $B_{eq}^{(H)} = \int_{-\infty}^{\infty}|\hat{H}(f)|^2\mathrm{d}f$ as follows

$$\mathcal{E}\left\{\xi(t)\right\} = \mu = (k\mathscr{T}B_{eq}^{(H)})$$

and

$$\gamma(0) = (k\mathscr{T}B_{eq}^{(H)})^2.$$

Using Equation 7 and Parseval's theorem the correlation time can be written as

$$\tau_0 = \frac{1}{(B_{eq}^{(H)})^2}\int_{-\infty}^{\infty}|\hat{H}(f)|^4\mathrm{d}f$$

from which we find that $\tau_0 \leq 1/B_{eq}^{(H)}$. In the case of dense sampling over a time period T, Equation 10 gives

$$\mathrm{var}(M) \leq \frac{(k\mathscr{T}B_{eq}^{(H)})^2}{TB_{eq}^{(H)}}.$$

In the case of sparse uncorrelated samples we have from Equation 6

$$\mathrm{var}(M) \approx \mu^2/N = (k\mathscr{T}B_{eq}^{(H)})^2/N.$$

3.1.3 Confidence Intervals

If the sampling distribution of the estimate M is known, it is possible to calculate the endpoints of an interval that with some specified probability contains the value of the population mean. Such a procedure is known as the method of confidence intervals.

Given a small number $\varepsilon > 0$ and confidence coefficient $p = 1-\varepsilon$, let $c_1(M)$ and $c_2(M)$ be the endpoints of an interval so that $\mathcal{P}\{c_1(M) \leq \mu \leq c_2(M)\} = p$. This is equivalent to the known probability $\mathcal{P}\{\eta_1(\mu) \leq M \leq \eta_2(\mu)\}$ if we choose $c_1(M) = \eta_2^{-1}(M)$ and $c_2(M) = \eta_1^{-1}(M)$.

For example, if a large number of independent and identically distributed random samples are used, M is approximately normal with mean μ and standard deviation $\sigma/\sqrt{N}$. For large N it follows that

$$\mathcal{P}\{M - \lambda_\varepsilon\frac{\sigma}{\sqrt{N}} < \mu < M + \lambda_\varepsilon\frac{\sigma}{\sqrt{N}}\} = \mathcal{P}\{\mu - \lambda_\varepsilon\frac{\sigma}{\sqrt{N}} < M < \mu + \lambda_\varepsilon\frac{\sigma}{\sqrt{N}}\} = p$$

where λ_ε is the 100ε % value of the standard normal deviate $Z = (M-\mu)/(\sigma/\sqrt{N})$, i.e.,

$$\varepsilon = \mathcal{P}\{|Z| > \lambda_\varepsilon\} = \frac{2}{\sqrt{2\pi}}\int_{\lambda_\varepsilon}^{\infty} e^{-t^2/2}\,\mathrm{d}t.$$

Now, for a given estimate M and confidence coefficient p we can determine the endpoints $M \pm \lambda_\varepsilon\sigma/\sqrt{N}$ of the interval that with probability $100p$ % contains μ.

As before, if the value of σ is not known and N is large, a common practice is to use the unbiased estimate obtained from Equation 8.

3.2 Cyclostationary Processes

Many nonstationary processes of interest in communications exhibit periodically time-variant statistics. Examples include pulse-amplitude, -width, and -position modulation, amplitude modulation, phase modulation, and periodic noise pulses. A process $\xi(t)$ is said to be *cyclostationary* if the mean and correlation are periodic. If the period is T we have

$$\begin{aligned} \mu(t) &= \mu(t+T) \\ r(t,s) &= r(t+T, s+T), \end{aligned} \tag{11}$$

and if $\int_0^T |\mu(t)|^2 \mathrm{d}t \ < \infty$, the mean can be represented as a Fourier series

$$\mu(t) = \sum_{n=-\infty}^{\infty} \mu_n e^{i2\pi nt/T}.$$

The Fourier coefficient μ_n is sometimes referred to as a *cyclic mean* and can be defined as

$$\mu_n = \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} \mu(t) e^{-i2\pi nt/T} \mathrm{d}t. \tag{12}$$

The covariance function $\gamma(t,s) = r(t,s) - \mu(t)\mu(s)$ is also periodic. It is sometimes useful to define the covariance in terms of *clock time* t and *lag* $\tau = s - t$ as follows

$$\tilde{\gamma}(t,\tau) = \gamma(t - \tau/2, t + \tau/2)$$

and if $\int_0^T |\tilde{\gamma}(t,\tau)|^2 \mathrm{d}t \ < \infty$ as a Fourier series

$$\tilde{\gamma}(t,\tau) = \sum_{n=-\infty}^{\infty} \tilde{\gamma}_n(\tau) e^{i2\pi nt/T}.$$

with Fourier coefficients

$$\tilde{\gamma}_n(\tau) = \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} \tilde{\gamma}(t,\tau) e^{-i2\pi nt/T} \mathrm{d}t. \tag{13}$$

As before, it is useful to define a correlation time. Since the process is not stationary, the average correlation time can be defined in terms of *clock time* averages as follows

$$\frac{\tau_0}{2} = \frac{\langle \int_0^\infty \tilde{\gamma}(t,\tau)\mathrm{d}\tau \rangle}{\langle \tilde{\gamma}(t,0) \rangle} = \frac{\int_0^\infty \tilde{\gamma}_0(\tau)\mathrm{d}\tau}{\tilde{\gamma}_0(0)}.$$

3.2.1 Dense Sampling

Equation 12 indicates that an estimate of the cyclic mean μ_n can be obtained from a sample function as follows

$$M_n(Z, \omega_1) = \frac{1}{Z} \int_{-Z/2}^{Z/2} \xi(t, \omega_1) e^{-i2\pi nt/T} \mathrm{d}t.$$

The expected value of the estimate is asymptotically unbiased

$$\mathcal{E}\left\{M_n(Z)\right\} = \mu_n + \sum_{k \neq n} \mu_k \operatorname{sinc}(\pi(k-n)Z/T) \to \mu_n \tag{14}$$

with variance

$$\operatorname{var}(M_n(Z)) = \frac{1}{Z^2} \iint_{-Z/2}^{Z/2} \gamma(t, s) e^{-i2\pi n(t-s)/T} \mathrm{d}t \, \mathrm{d}s. \tag{15}$$

If $\lim_{Z\to\infty} \operatorname{var}(M_n(Z)) = 0$, then the process is mean-square *cycloergodic* at the frequency n/T.

3.2.2 Sparse Uncorrelated Samples

Suppose that $\xi(t)$ is cycloergodic at all frequencies of interest and that $\gamma(t, s) \approx 0$ when $|s - t| \geq \tau_0$. We can then use approximately uncorrelated samples to obtain an estimate of the cyclic mean as follows

$$M_n = \frac{1}{N} \sum_{k=0}^{N-1} \xi(t_k) e^{-i2\pi nt_k/T}$$

where $|t_k - t_{k\pm1}| \geq \tau_0$. If we let

$$t_k = kT(J + \varepsilon) \tag{16}$$

where $TJ \geq \tau_0$, for some positive integer J, and $0 < \varepsilon < 1$, the estimate can be written as

$$M_n = \frac{1}{N} \sum_{k=0}^{N-1} \xi(t_k) e^{-i2\pi nk\varepsilon}$$

with expected value

$$\mathcal{E}\left\{M_n\right\} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=-\infty}^{\infty} \mu_j \, e^{i2\pi(j-n)k\varepsilon} = \sum_{j\varepsilon\in\mathbb{Z}} \mu_{j+n} + \sum_{j\varepsilon\notin\mathbb{Z}} \mu_{j+n} \frac{1}{N} \frac{e^{i2\pi Nj\varepsilon} - 1}{e^{i2\pi j\varepsilon} - 1}.$$

In the limit we find that the estimate is in general biased

$$\lim_{N\to\infty} \mathcal{E}\left\{M_n\right\} = \sum_{j\varepsilon\in\mathbb{Z}} \mu_{j+n}.$$

However, if ε is irrational or if $\mu(t)$ is bandlimited so that $\mu_{j+n} = 0,\ |j| \geq \lfloor 1/\varepsilon \rfloor$, the estimate is asymptotically unbiased.

For practical applications, we assume that the increment is a small rational number which can be written as $\varepsilon = 1/K$, where K is a positive integer. For this case, the limiting expected value of the estimate can also be written as a discrete Fourier transform

$$\lim_{N\to\infty} \mathcal{E}\{M_n\} = \frac{1}{K}\sum_{k=0}^{K-1} \mu(kT/K)e^{i2\pi nk/K}.$$

We see that, for sparse uncorrelated samples, the expected value of M_n is equivalent to estimating μ_n using the discrete Fourier transform as an approximation to Equation 12. It should be noted that we must also have $K >> n$ when estimating higher order cyclic means.

The variance of the estimate is

$$\mathrm{var}(M_n) = \frac{1}{N^2}\sum_{k=0}^{N-1} \gamma(t_k, t_k). \tag{17}$$

Since the covariance is periodic, we can use Equation 16 to obtain

$$\frac{1}{N}\sum_{k=0}^{N-1} \gamma(kT\varepsilon, kT\varepsilon) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{\gamma}(kT\varepsilon, 0).$$

As before, if we let $K = 1/\varepsilon$ and N is large, we need only average the covariance function over a period and therefore

$$\frac{1}{N}\sum_{k=0}^{N-1} \gamma(kT\varepsilon, kT\varepsilon) \to \frac{1}{K}\sum_{k=0}^{K-1} \tilde{\gamma}(kT/K, 0).$$

Note that $\frac{1}{K}\sum_{k=0}^{K-1} \tilde{\gamma}(kT/K, 0)$ is the discrete Fourier transform approximation of $\tilde{\gamma}_0(0)$ (see Equation 13) so that when ε is sufficiently small the variance can be approximated as

$$\mathrm{var}(M_n) \approx \frac{\tilde{\gamma}_0(0)}{N}. \tag{18}$$

This result is similar to that obtained for uncorrelated samples of a stationary process (Equation 6) with the zero-lag stationary process covariance function replaced by the zero-lag time average of the cyclostationary process covariance function.

As with stationary processes, calculating the uncertainty in the sparse sampling estimate requires additional knowledge of the process, in this case $\tilde{\gamma}_0(0)$, which is often not directly available. When N is large, $\tilde{\gamma}_0(0)$ can be replaced by an estimate of the form

$$\tilde{S}_0(0) = \frac{1}{N-1}\sum_{k=0}^{N-1} |\xi(t_k) - M_0|^2.$$

3.2.3 Random Sampling

In this subsection we will show that a cyclostationary process can be converted into a stationary process by random sampling. The average of the stationary process is an unbiased estimate of μ_0 and the results of Section 3.1 can be used to calculate uncertainties for this estimate.

By random sampling, we mean that the observation times are random and uniformly distributed over a period T. This gives a new process defined as follows

$$X(t) = \xi(t+\theta), \quad \theta \text{ uniform over } [0,T).$$

The new process is stationary since the mean is constant

$$\mathcal{E}\{X(t)\} = \mathcal{E}\{\mathcal{E}\{\xi(t+\theta)|\,\theta\}\} = \frac{1}{T}\int_0^T \mu(\theta)\mathrm{d}\theta = \mu_0 \tag{19}$$

and the correlation function depends only on the lag $\tau = s - t$

$$\begin{aligned}\mathcal{E}\{X(t)X^*(s)\} &= \mathcal{E}\{\mathcal{E}\{\xi(t+\theta)\xi^*(s+\theta)|\,\theta\}\} \\ &= \frac{1}{T}\int_0^T r(t+\theta, s+\theta)\mathrm{d}\theta \\ &= \frac{1}{T}\int_0^T r(\theta, s-t+\theta)\mathrm{d}\theta = r(\tau).\end{aligned}$$

Note that the mean and correlation functions are time averages of the cyclostationary statistics.

The covariance function can be written in terms of the correlation function and the mean as follows

$$\gamma(\tau) = r(\tau) - \mu_0^2.$$

3.2.4 An Example – Pulse Modulation

In this subsection, we will calculate the variance of the mean power estimator for a baseband pulse modulated process using the different sampling schemes described above. The process used in this example has the form

$$\hat{n}(t) = \sum_{n=-\infty}^{\infty} Z_n\, p\,(t - nT)$$

where $p(t)$ is real and is zero outside of the interval $[0,T)$ and Z_n are iid complex random variables. The *instantaneous* power can be written as

$$\xi(t) = \sum_{n=-\infty}^{\infty} W_n\, p^{\,2}(t - nT)$$

where $W_n = |Z_n|^2$.

To calculate the variance of the estimator we first need to obtain expressions for the process mean and correlation functions. Using the notation $\mu_w = \mathcal{E}\{W_n\}$ and $\sigma_w^2 = \mathcal{E}\{W_n^2\} - \mu_w^2$, we have

$$\mathcal{E}\{W_n W_m\} = \sigma_w^2 \delta_{nm} + \mu_w^2. \tag{20}$$

The process mean is simply

$$\mu(t) = \mathcal{E}\{\xi(t)\} = \mu_w \sum_{n=-\infty}^{\infty} p^2(t - nT).$$

and the correlation function is

$$r(t,s) = \mathcal{E}\{\xi(t)\xi(s)\} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{E}\{W_n W_m\}\ p^2(t - nT)\ p^2(s - mT).$$

Using Equation 20 gives

$$r(t,s) = \mu(t)\mu(s) + \sigma_w^2 \sum_{n=-\infty}^{\infty} p^2(t - nT) p^2(s - nT)$$

and hence

$$\gamma(t,s) = \sigma_w^2 \sum_{n=-\infty}^{\infty} p^2(t - nT) p^2(s - nT).$$

Clearly, this process is cyclostationary and has the periodicity as described in Equation 11.

We now assume that the estimator is obtained from a densely sampled record of length Z and calculate the variance of the estimator. To minimize the bias (Equation 14) we require $Z >> T$. It is useful to approximate Equation 15 in terms of the minimum number of intervals of length T in the record. Using the floor function notation define

$$2N = \left\lfloor \frac{Z}{T} \right\rfloor$$

and hence

$$\mathrm{var}(M_n(N)) \approx \frac{\sigma_w^2}{(2NT)^2} \sum_{m=-\infty}^{\infty} \left| \int_{-NT}^{NT} p^2(t - mT) e^{-i2\pi nt/T} \mathrm{d}t \right|^2.$$

We can sum the series (approximately) and write the expression for the variance in terms of the Fourier transform of the pulse squared

$$\mathrm{var}(M_n(N)) \approx \frac{\sigma_w^2}{2NT^2} \left| \int_{-\infty}^{\infty} p^2(t) e^{-i2\pi nt/T} \mathrm{d}t \right|^2.$$

Applying the commonly used rectangular pulse

$$p(t) = p_{rect}\,(t) = \begin{cases} 1 & 0 \le t < T_0,\ T_0 < T \\ 0 & \text{else} \end{cases} \tag{21}$$

gives

$$\text{var}(M_n(N)) \approx \frac{\sigma_w^2 T_0^2}{2NT^2}\,\text{sinc}^2(\pi n T_0/T). \tag{22}$$

Next we will assume that the estimator is obtained from N sparse uncorrelated samples. Using Equation 17 gives

$$\text{var}(M_n) = \frac{\sigma_w^2}{N^2}\sum_{k=0}^{N-1}\sum_{n=-\infty}^{\infty} p^{\,4}(kT\varepsilon - nT).$$

For $N >> K$ and $\varepsilon = 1/K << 1$, we can use Equation 18 and Equation 13 to obtain

$$\text{var}(M_n) \approx \frac{\tilde{\gamma}_0(0)}{N} = \frac{\sigma_w^2}{NT}\int_0^T p^{\,4}(t)\mathrm{d}t.$$

For the rectangular pulse of Equation 21

$$\text{var}(M_n(N)) \approx \frac{\sigma_w^2 T_0}{NT}. \tag{23}$$

So far we have compared the variance of the mean estimator when dense and sparse uncorrelated sampling schemes are used. Often we only need to estimate the time average mean power μ_0. Such an estimate is readily obtained by using random sampling. In what follows, we will calculate the variance of the estimator M_0 obtained from random samples of the pulse modulated processes.

We begin with the modified pulse modulated process

$$X(t) = \sum_{n=-\infty}^{\infty} Z_n\, p^2\,(t - nT + \theta)$$

where θ is uniformly distributed over the period T. Using the results of Section 3.2.3 we can calculate the process mean and correlation, viz.,

$$\mathcal{E}\,\{X(t)\} = \frac{\mu_w}{T}\int_{-\infty}^{\infty} p^2(\theta)\mathrm{d}\theta = \mu_0$$

and the correlation as a function of lag $\tau = s - t$

$$r(\tau) = r_1(\tau) + r_2(\tau)$$

where

$$\begin{aligned} r_1(\tau) &= \sigma_w^2 \sum_{n=-\infty}^{\infty} \mathcal{E}\left\{p^{\,2}(t+\theta-nT)p^{\,2}(s+\theta-nT)\right\} \\ r_2(\tau) &= \mu_w^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{E}\left\{p^{\,2}(t+\theta-nT)p^{\,2}(s+\theta-mT)\right\}. \end{aligned}$$

These equations reduce to

$$\begin{aligned} r_1(\tau) &= \frac{\sigma_w^2}{T} \int_{-\infty}^{\infty} p^{\,2}(\theta)p^{\,2}(\tau+\theta)\mathrm{d}\theta \\ r_2(\tau) &= \frac{\mu_w^2}{\sigma_w^2} \sum_{n=-\infty}^{\infty} r_1(\tau-nT). \end{aligned}$$

Note that $r_2(\tau)$ is a scaled periodic extension of $r_1(\tau)$. Using $p_{rect}(t)$ defined by Equation 21 gives

$$r_1(\tau) = \begin{cases} \sigma_w^2(T_0-|\tau|)/T & |\tau| \le T_0 \\ 0 & \text{else} \end{cases}$$

i.e., the correlation function consists of a single triangle function of height $\sigma_w^2 T_0/T$ and base $2T_0$ centered at $\tau = 0$ and the periodic extension of a triangle function of height $\mu_w^2 T_0/T$ and base $2T_0$. The covariance function is

$$\gamma(\tau) = r(\tau) - \mu_w^2 \left(\frac{T_0}{T}\right)^2.$$

Clearly, we have

$$\lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} \gamma(\tau)\mathrm{d}\tau = 0$$

so the process is mean-square ergodic. However, since the covariance function itself does not tend to a limit, the time average does not tend to the mean with probability one.

If we choose the special case $T_0 = T$ then

$$\begin{aligned} r_1(\tau) &= \begin{cases} \sigma_w^2(1-|\tau|/T) & |\tau| \le T \\ 0 & \text{else} \end{cases} \\ r_2(\tau) &= \mu_w^2 \end{aligned}$$

The process correlation time is $\tau_0 = T$ and the time average tends to the mean with probability one. If we obtain N uncorrelated samples at time intervals of τ_0 (or greater)the variance of the estimate is

$$\mathrm{var}(M) = \frac{\sigma_w^2}{N}.$$

In the case of dense sampling over the same time window (i.e., $T_N = N\tau_0$), we have (Equation 9)

$$\mathrm{var}(M) = \frac{2\sigma_w^2}{T_N} \int_0^{T_N} \left(1 - \frac{\tau}{T_N}\right) \frac{T - \tau}{T} \, \mathrm{d}\tau = \frac{2\sigma_w^2}{N} \int_0^1 \left(1 - \frac{\tau}{N}\right) (1 - \tau) \, \mathrm{d}\tau$$

which reduces to

$$\mathrm{var}(M) = \frac{\sigma_w^2}{N} \left(1 - \frac{1}{3N}\right).$$

Hence even for small N, the variance is essentially the same as that for sparse uncorrelated samples using a sample spacing of τ_0.

It is useful to compare this result with the estimator variance given in Equation 22 and Equation 23 for M_0 and $T_0 = T$. Clearly, the random sampling variance and the sparse sampling variance are the same (assuming N is large). To compare this with the variance of the dense sampling estimator we note that $2N$ in Equation 22 is essentially the same as T_N/T given above. If the sample spacing is equal to the correlation time $\tau_0 = T$ we find that the variance is approximately the same for all the sampling schemes. It should be noted, however, that the random sampling estimator is unconditionally unbiased (Equation 19) while the other estimators are only asymptotically unbiased.

4. ESTIMATION OF THE AMPLITUDE PROBABILITY DENSITY

Quantile statistics play an important role in characterizing radio noise processes and in predicting the performance of radio receivers. Typically, radio engineers are interested in the percent of time that noise amplitudes exceed a specified threshold. The threshold may represent, for example, the point at which the symbol error rate exceeds the capability of error correction algorithms. Radio engineers often refer to such statistics as *amplitude probability distributions*. The APD is the compliment of the first-order distribution function of the complex baseband noise envelope expressed in decibels. In this section, we define the APD estimator and derive expressions for calculating quantile confidence intervals.

An estimate of the APD can be obtained from an experiment which yields discrete data samples from a complex baseband noise process $\hat{n}(t, \omega)$. Here, we assume that the samples used to construct the APD are approximately independent. This is usually accomplished in practice by designing the experiment so that the time between data samples exceeds the process correlation time. We have then a random sample consisting of a sequence of N observed values of the noise envelope

$$\xi_k(\omega_1) = |\hat{n}(t_k, \omega_1)| \qquad k = 1, 2, \cdots, N \qquad |t_{k+1} - t_k| \geq \tau_0.$$

If the process is stationary and ergodic, the ξ_k can be considered as iid random variables with distribution $F(x)$ and $\mathcal{P}\{\xi_k > x\} = 1 - F(x)$.

If we denote by ν the number of sample values that are greater than some specified value x, we obtain a step-function estimate of the APD with an incremental step height of $1/N$. The probability that x is exceeded is simply the frequency of the event $\{\xi_k > x\}$ in the sequence of N observations. In practice, the APD estimate is easily calculated by ordering the samples $\xi[1] \geq \xi[2] \geq \cdots \geq \xi[N]$ where $\xi[\nu]$ denotes the ν^{th} order-statistic and assigning the probabilities

$$\mathcal{P}\{\xi_k > \xi[\nu]\} = \frac{\nu}{N}.$$

Obviously the total number of samples needed is predicated on the desired accuracy of the probability estimate. This will, however, vary with the quantile, being most accurate for the median and least accurate for the tails. For example, it would seem quite unlikely that $\mathcal{P}\{\xi_k > \xi[N]\} = 1$ which indicates that the endpoints of the APD estimate are dubious as is shown in the quantitative analysis which follows.

4.1 Probability Coordinates for the APD

Since the amplitude of a complex Gaussian process (e.g., radio receiver noise) is Rayleigh distributed, the usual engineering practice is to graph APDs on probability paper that linearizes Rayleigh distributed amplitudes. This allows the engineer to determine, at a glance, if the measured process is Rayleigh distributed and if not, how and why it deviates from a Rayleigh distribution. To determine the scales for Rayleigh graph paper we begin

with the distribution of the amplitude (squared) of a stationary complex Gaussian process

$$\mathcal{P}\{|\hat{n}(t)|^2 > v^2\} = e^{-v^2/v_0^2}$$

where $v_0^2 = \mathcal{E}\{|\hat{n}(t)|^2\}$. Engineers prefer quantities in decibels, e.g., $W(t) = 20\log|\hat{n}(t)|$ in which case the distribution becomes

$$\mathfrak{P} = \mathcal{P}\{W(t) > w\} = e^{-10^{w/10}/v_0^2}.$$

If we take for the abscissa scale $x = -10\log\ln 1/\mathfrak{P}$ and the ordinate scale $y = w$ we have the desired linear relationship $x = -y + w_0$, where $w_0 = 20\log v_0$.

4.2 Accuracy of the APD Estimate

As in the case of the mean, it is important to make quantitative remarks about the accuracy of the APD estimate. In this section we will discuss both convergence of the estimate and a method for calculating confidence intervals (ref [4]). It is important to note that the following concepts require that the data ξ_k are iid observations.

To begin, we define the quantile x_p as

$$\mathcal{P}\{\xi_k > x_p\} = 1 - F(x_p) = p$$

and the related random variables

$$\eta_k = \begin{cases} 1 & \text{if } \xi_k > x_p \\ 0 & \text{else} \end{cases} \qquad \text{and} \qquad \nu = \sum_{k=1}^{N} \eta_k.$$

The random variable ν is just the number of sample values greater than x_p in N observations.

The probability that exactly m observations are greater than x_p and $N - m$ observations are less than or equal to x_p is $p^m q^{N-m}$ where $q = 1 - p$. The number of such sample sets is

$$\binom{N}{m} = \frac{N!}{m!(N-m)!}$$

hence, the probability that a sample set contains exactly m observations greater than x_p, irrespective of the order, is

$$\mathcal{P}\{\nu = m; N\} = \binom{N}{m} p^m q^{N-m}.$$

By *De Moivre's Theorem* [5] ν/N is asymptotically normal with $\mathcal{E}\{\nu/N\} = p$ and $\text{var}(\nu/N) = p\,q/N$. We see immediately that when the ξ_k are iid, as $N \to \infty$, the estimate converges in probability, i.e., $\mathcal{P}\{|\nu/N - p| < \epsilon\} = 1$ for any $\epsilon > 0$.

To calculate confidence intervals for a population quantile x_p we can use the following relations

$$\mathcal{P}\{\nu \geq m; N\} = \textstyle\sum_{k=m}^{N} \binom{N}{k} p^k q^{N-k} = b(m)$$
$$\mathcal{P}\{\nu < m; N\} = 1 - b(m).$$

Now let m_- and m_+ be the indices for two ordered samples so that $\xi[m_-] > \xi[m_+]$. Then it follows that

$$\mathcal{P}\{\xi[m_-] \geq x_p \geq \xi[m_+]; m_+ > m_-\} = \mathcal{P}\{\nu \geq m_-\} - \mathcal{P}\{\nu \geq m_+\} = b(m_-) - b(m_+)$$

hence, we can say that with $(b(m_-) - b(m_+)) \cdot 100\%$ confidence the population quantile x_p is in the interval $[\xi[m_+], \xi[m_-]\,]$

For example, given $N \geq 2$ samples, the probability that the median falls between the smallest and the largest sample is given by

$$\mathcal{P}\{\xi[1] \geq \theta_{1/2} \geq \xi[N]\} = 1 - \frac{1}{2^{N-1}}.$$

4.2.1 Approximation for Large N

If $Npq >> 1$ and $|m - Np|^3/(Npq)^2 << 1$ the following approximation for the binomial distribution can be used [6]

$$\mathcal{P}\{\nu \geq m\} = \sum_{k=m}^{N} \binom{N}{k} p^k q^{N-k} \rightarrow 1 - \Phi\left(\frac{m - Np}{\sqrt{Npq}}\right)$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \mathrm{d}t.$$

We can now specify a desired confidence $c \cdot 100\%$ and calculate the indices of the order statistics that bound the confidence interval. Obviously we require that

$$b(m_-) - b(m_+) = c.$$

In addition we want the mean value of ν (Np) for the quantile x_p to be near the center of the interval. This is accomplished by setting $b(m_+) + c/2 = 1/2$ or

$$b(m_+) = (1 - c)/2.$$

Applying these two conditions gives the indices of the order statistics that bound the confidence interval

$$\begin{aligned} m_+ &= \lceil Np + \sqrt{Npq}\ \Phi^{-1}\left(\tfrac{1+c}{2}\right) \rceil \\ m_- &= \lfloor Np - \sqrt{Npq}\ \Phi^{-1}\left(\tfrac{1+c}{2}\right) \rfloor\,. \end{aligned}$$

The median m_0 is given by $\mathcal{P}\{\xi[m_0] < x_{1/2}\} = 1/2$, hence $m_0 = [N/2]$.

As an example, assume that we wish to find the 90% confidence interval for the median given 2000 iid samples. In this case, $\Phi^{-1}\left(\frac{1+c}{2}\right) = 1.64$ and hence,

$$\begin{aligned} m_+ &= \lceil (1000 + 36.7) \rceil = 1037 \\ m_- &= \lfloor (1000 - 36.7) \rfloor = 963 \\ m_0 &= 1000 \end{aligned}$$

therefore

$$\mathcal{P}\{X[963] \geq x_{1/2} \geq X[1037]\} \geq 0.9\,.$$

5. SECOND-ORDER STATISTICS

Second-order statistics provide information about the statistical dependence of a noise process at two different times. This information can be used, for example, to predict the duration of undesirable correlated events and for the development of algorithms to mitigate these effects. Fourier transforms of second-order statistics provide information about the characteristics of the frequency content of the process.

An important and easily measured second-order statistic is the covariance. Specifically, the covariance function gives a measure of the correlation of the process at two different times. If the process is stationary, the covariance only depends on the time difference. Furthermore, if all of the joint distribution functions of the process are normal, the covariance and the mean are all that is required to completely characterize the process. If the process is stationary, the Fourier transform of the covariance function is

$$\Gamma(f) = \int_{-\infty}^{\infty} \gamma(\tau) e^{-i2\pi f\tau} \mathrm{d}\tau \,,$$

which evidently shows how the mean power of the process is distributed over frequency, e.g.,

$$\gamma(0) = \int_{-\infty}^{\infty} \Gamma(f) \mathrm{d}f.$$

Hence, the function $\Gamma(f)$ is usually referred to as the power spectrum.

In this section we will describe covariance and power spectrum estimators that are commonly applied to noise measurement data. Mathematical expressions for the expected value and covariance of these estimators will be given. Finally, we will obtain asymptotic statistical distributions for discrete estimates of the power spectrum based on the discrete Fourier Transform.

Before proceeding, it is useful to define some standard notation used in the remainder of this report. First, the notation $\mathrm{N}(\mu, \sigma^2)$ used in conjunction with a real random variable ξ means that ξ is normally distributed with mean μ and variance σ^2. If ξ is complex normal we will use the notation $\mathrm{N}^c(\mu, \sigma^2)$, where in general the mean is complex. Note that for identically distributed independent complex random variables, $\mathrm{Re}\,\xi$ and $\mathrm{Im}\,\xi$ are $\mathrm{N}(\mathrm{Re}\,\mathcal{E}\{\xi\}, \mathrm{var}(\xi)/2)$ and $\mathrm{N}(\mathrm{Im}\,\mathcal{E}\{\xi\}, \mathrm{var}(\xi)/2)$. We will also adopt the standard notation χ^2_n to represent a *chi-square* random variable with n degrees of freedom, i.e., if ξ_k are a sequence of real $N(0,1)$ independent random variables,

$$\chi^2_n = \sum_{k=1}^{n} \xi_k^2. \tag{24}$$

Finally, the symbol Γ_n will be used to denote the discrete Fourier transform of the covariance function defined as

$$\Gamma_n = \sum_{k=0}^{N-1} \gamma_k e^{-i2\pi nk/N}.$$

5.1 Estimation of the Covariance Function

Suppose that $\xi(t,\omega)$ is a zero mean complex baseband stationary and ergodic noise process. A commonly used method for estimating $\gamma(\tau) = \mathrm{cov}(\xi(t), \xi(t+\tau))$ is to obtain a sample function $\xi(t,\omega_1)$ of time duration T and calculate the time average

$$c(\tau,\omega_1) = \begin{cases} \frac{1}{T}\int_0^{T-\tau} \xi^*(t,\omega_1)\xi(t+\tau,\omega_1)\mathrm{d}t & \text{if } 0 \le \tau \le T \\ 0 & \text{if } \tau > T \end{cases} \tag{25}$$

and $c(-\tau) = c^*(\tau)$. This estimator is a random process, hence, we need to determine its mean and covariance.

Since the underlying process is stationary, the mean of the estimator is easily calculated

$$\mathcal{E}\left\{c(\tau)\right\} = \frac{1}{T}\int_0^{T-\tau} \mathcal{E}\left\{\xi^*(t)\xi(t+\tau)\right\}\mathrm{d}t = \begin{cases} \gamma(\tau)\left(1-\frac{\tau}{T}\right) & \text{if } 0 \le \tau \le T \\ 0 & \text{if } \tau > T \end{cases}$$

and $\mathcal{E}\left\{c(-\tau)\right\} = \mathcal{E}\left\{c^*(\tau)\right\}$. This result shows that the estimator is biased, however it is asymptotically unbiased (i.e., $\mathcal{E}\left\{\tau\right\} \to \gamma(\tau)$ as $T \to \infty$).

Calculating the covariance of the estimator is somewhat involved and is formally obtained from the following expression

$$\mathrm{cov}(c(\tau_1), c(\tau_2)) = \frac{1}{T^2}\int_0^{T-\tau_1}\int_0^{T-\tau_2} \mathrm{cov}\left(\xi^*(t)\xi(t+\tau_1), \xi^*(s)\xi(s+\tau_2)\right)\mathrm{d}t\,\mathrm{d}s$$

with $0 \le \tau_1 \le T$ and $0 \le \tau_2 \le T$.

To simplify this expression, the usual procedure is to invoke the fourth-order cumulant $\kappa = \mathrm{cum}(\xi_1, \xi_2, \xi_3^*, \xi_4^*)$, and apply the following formula

$$\mathrm{cov}(\xi_1\xi_2, \xi_3\xi_4) = \mathrm{cov}(\xi_1, \xi_3)\mathrm{cov}(\xi_2, \xi_4) + \mathrm{cov}(\xi_1, \xi_4)\mathrm{cov}(\xi_2, \xi_3) + \kappa.$$

It is a common practice to assume that the contribution due to κ is small and can be ignored [7]. This assumption is apparently based on the fact that $\kappa = 0$ for normal processes and that the contribution due to κ can be neglected when $\xi(t)$ is a non-Gaussian linear process. If the contribution of the fourth order cumulant is neglected, we obtain

$$\begin{aligned} \mathrm{cov}(c(\tau_1), c(\tau_2)) &= \frac{1}{T^2}\int_0^{T-\tau_1}\int_0^{T-\tau_2} \Big[\mathcal{E}\left\{\xi^*(t)\xi(s)\right\}\mathcal{E}\left\{\xi(t+\tau_1)\xi^*(s+\tau_2)\right\} \\ &+ \mathcal{E}\left\{\xi^*(t)\xi^*(s+\tau_2)\right\}\mathcal{E}\left\{\xi(t+\tau_1)\xi(s)\right\}\Big]\mathrm{d}t\,\mathrm{d}s. \end{aligned} \tag{26}$$

If T is large, the following approximate result can be used [7]

$$\mathrm{cov}(c(\tau_1), c(\tau_2) \approx \frac{1}{T}\int_{-\infty}^{\infty} \Big[\gamma^*(t)\gamma(t+\tau_2-\tau_1) + \gamma'^*(t+\tau_2)\gamma'(t-\tau_1)\Big]\mathrm{d}t$$

where $\gamma'(\tau) = \mathcal{E}\{\xi(t)\xi(t+\tau)\}$. For a complex baseband stationary process, $\gamma'(\tau) = 0$ and hence

$$\mathrm{cov}(c(\tau_1), c(\tau_2) \approx \frac{1}{T}\int_{-\infty}^{\infty} \gamma^*(t)\gamma(t+\tau_2-\tau_1)\mathrm{d}t.$$

This result shows that for ergodic processes, the variance of the estimator decreases as the record length increases.

In practice, the sample function observations yield a discrete time sequence ξ_k. The corresponding discrete estimator of the covariance function is given by

$$c_m = \frac{1}{N}\sum_{k=0}^{N-m-1} \xi_k^* \xi_{k+m} \qquad m = 0, 1, \cdots, N-1.$$

The covariance of the estimator is approximately

$$\mathrm{cov}(c_m, c_n) \approx \frac{1}{N}\sum_{k=-\infty}^{\infty} \{\gamma_k^* \gamma_{k+n-m}\}$$

where now the process covariance function is discrete and is defined as a sequence

$$\gamma_m = \mathcal{E}\{\xi_k^* \, \xi_{k+m}\}.$$

5.2 Estimation of the Power Spectrum

The estimate of the power spectrum is obtained by taking the Fourier transform of a sample function. Assuming $\xi(t)$ is a zero mean sample function we have

$$C(f) = \frac{1}{T}\left|\int_0^T \xi(t)e^{-i2\pi ft}\mathrm{d}t\right|^2 = \frac{1}{T}\int_0^T\int_0^T \xi(t)\xi^*(s)e^{-i2\pi f(t-s)}\mathrm{d}t\,\mathrm{d}s.$$

A change of variables $u = t - s$ and $v = s$ yields

$$C(f) = \int_0^T \left[\frac{1}{T}\int_0^{T-u} \xi^*(v)\xi(v+u)\mathrm{d}v\right] e^{-i2\pi fu}\mathrm{d}u + \int_{-T}^0 \left[\frac{1}{T}\int_{-u}^T \xi^*(v)\xi(v+u)\mathrm{d}v\right] e^{-i2\pi fu}\mathrm{d}u.$$

Using the definition of $c(u)$ from Equation 25, the first integral above is simply

$$\int_0^T c(u)e^{-i2\pi fu}\mathrm{d}u.$$

After a change of variables, the second integral can be written as

$$\int_{-T}^0 \left[\frac{1}{T}\int_0^{T-|u|} \xi^*(v+|u|)\xi(v)\mathrm{d}v\right] e^{-i2\pi fu}\mathrm{d}u = \int_{-T}^0 c^*(|u|)e^{-i2\pi fu}\mathrm{d}u.$$

Recalling that $c^*(|u|) = c(-|u|)$ gives

$$C(f) = \int_{-T}^{T} c(u)e^{-i2\pi fu}\mathrm{d}u$$

which is the Fourier transform of the covariance function estimator. Thus the covariance function estimator and the sample spectrum are Fourier transform pairs

$$C(f) = \int_{-T}^{T} c(u)e^{-i2\pi fu}\mathrm{d}u \qquad c(u) = \int_{-\infty}^{\infty} C(f)e^{i2\pi fu}\mathrm{d}f.$$

The expected value of the spectral estimator of an ergodic stationary process is readily calculated as

$$\mathcal{E}\left\{C(f)\right\} = \int_{-T}^{T} \mathcal{E}\left\{c(u)\right\} e^{-i2\pi fu}\mathrm{d}u = \int_{-T}^{T} \gamma(u)\left(1 - \frac{|u|}{T}\right) e^{-i2\pi fu}\mathrm{d}u.$$

Hence, the spectral estimator is asymptotically unbiased.

Since the measured data is usually digitized we need to examine the properties of the spectral estimator based on the discrete Fourier Transform. For equally spaced samples ξ_k and spacing increment Δt, the discrete version of the spectral estimator is

$$C(f_n) = C_n = \frac{\Delta t}{N}\left|\sum_{k=0}^{N-1} \xi_k e^{-i2\pi nk/N}\right|^2 \tag{27}$$

where $f_n = n/(N\Delta t)$. In the sequel, we will refer to the C_n as spectral components.

5.2.1 Statistical Characteristics of the Discrete Spectral Estimate

The expected value of the discrete estimator is readily calculated as follows

$$\mathcal{E}\left\{C_n\right\} = \Delta t \sum_{m=-N}^{N-1}\left(\frac{N - |m|}{N}\right)\gamma_m\, e^{-i2\pi nm/N}$$

which is asymptotically unbiased.

We have seen that if the process is ergodic, the covariance function estimator converges to the covariance function as the data record length increases. This is not true, however, for the spectral estimator. This state of affairs can readily be demonstrated when the ξ_k are real iid normally distributed samples and the spectral estimator is calculated via the discrete Fourier transform.

We begin by assuming that each sample is $N(0,\sigma^2)$ and consider the expected values

$$\mathcal{E}\{C_n\} = \frac{\Delta t}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1}\mathcal{E}\{\xi_k\xi_m\}\,e^{i2\pi n(k-m)/N} = \sigma^2\Delta t$$

and

$$\mathcal{E}\{C_{n_1}C_{n_2}\} = \frac{(\Delta t)^2}{N^2}\sum_{j,k,\ell,m}\mathcal{E}\{\xi_j\xi_k\xi_\ell\xi_m\}\,e^{i2\pi(n_1(j-k)+n_2(\ell-m))/N}.$$

Since the process is iid normal, the expectation of the four samples is zero when the indices are all different. Futhermore $\mathcal{E}\{\xi_k^4\} = \sigma^4$ and $\mathcal{E}\{\xi_j\xi_j\xi_k\xi_k\} = \mathcal{E}\left\{\xi_j^2\right\}\mathcal{E}\{\xi_k^2\} = \sigma^4$ when $k \neq j$. This can be written in terms of Kronecker deltas as follows

$$\mathcal{E}\{\xi_j\xi_k\xi_\ell\xi_m\} = \sigma^4(\delta_{jk}\delta_{\ell m} + \delta_{j\ell}\delta_{km} + \delta_{jm}\delta_{k\ell})$$

and hence

$$\mathcal{E}\{C_{n_1}C_{n_2}\} = \frac{\sigma^4(\Delta t)^2}{N^2}\left(N^2 + \sum_{j=0}^{N-1}\sum_{k=0}^{N-1}e^{i2\pi(j-k)(n_1+n_2)/N} + \sum_{\ell=0}^{N-1}\sum_{m=0}^{N-1}e^{i2\pi(\ell-m)(n_1-n_2)/N}\right)$$

or

$$\mathcal{E}\{C_{n_1}C_{n_2}\} = \sigma^4(\Delta t)^2\left[1 + \left(\frac{\sin\pi(n_1+n_2)}{N\sin\pi(n_1+n_2)/N}\right)^2 + \left(\frac{\sin\pi(n_1-n_2)}{N\sin\pi(n_1-n_2)/N}\right)^2\right].$$

The covariance is obtained from $\mathrm{cov}(C_{n_1}, C_{n_2}) = \mathcal{E}\{C_{n_1}C_{n_2}\} - \mathcal{E}\{C_{n_1}\}\mathcal{E}\{C_{n_1}\}$ yielding

$$\mathrm{cov}(C_{n_1}, C_{n_2}) = \sigma^4(\Delta t)^2\left[\left(\frac{\sin\pi(n_1+n_2)}{N\sin\pi(n_1+n_2)/N}\right)^2 + \left(\frac{\sin\pi(n_1-n_2)}{N\sin\pi(n_1-n_2)/N}\right)^2\right]$$

and

$$\mathrm{cov}(C_{n_1}, C_{n_2}) = \begin{cases} \sigma^4(\Delta t)^2 & n_1 = n_2 \\ 0 & n_1 \neq n_2 \end{cases}.$$

This result demonstrates that the variance of the estimator does not decrease with the number of samples. Furthermore, adjacent samples are uncorrelated and, for a fixed record length, as N increases, the time between uncorrelated samples decreases. As a consequence, the rapidity of the fluctuations of the estimator increase and the variance does not decrease.

General results for the variance have been obtained for non-Gaussian non-white stationary processes with specific *mixing* assumptions (i.e., ξ_k and ξ_ℓ become statistically independent as $|k-\ell| \to \infty$) [8]. For example,

$$\mathrm{cov}(C_{n_1}, C_{n_2}) = (\Delta t\Gamma_{n_1})^2\delta_{n_1n_2} + O(N^{-1}).$$

5.2.2 Averaged Spectral Estimates

A variety of methods have been developed to mitigate the unwieldy character of the spectral estimator described above. In particular, a simple average of independent spectral components is an effective way to reduce the variance. This is sometimes called the *Welch Method* [9]. The asymptotic statistical characteristics of such an estimator are described below.

We begin with the Fourier transform of a *windowed* segment of data

$$X_n = \sum_{k=0}^{N-1} w(k/N)\xi_k e^{-i2\pi kn/N}.$$

Here, it is assumed that the window, $w(t)$, is of bounded variations and $w(t) = 0$ for $t < 0$ and $t > 1$. Also, the ξ_k are complex samples of a stationary zero-mean noise process with mixing properties as described in [8].

According to a central limit theorem for time series [8] the complex random variables X_n are asymptotically independent $\mathrm{N}^c(0, N\beta_n)$ variates where $\beta_n = \Gamma_n \int w^2(t)\mathrm{d}t$. The $|X_n|^2$ are asymptotically independent $N\beta_n\chi_2^2/2$ random variables (see Equation 24) with $\mathcal{E}\{|X_n|^2\} = N\beta_n$ and $\mathrm{var}(|X_n|^2) = (N\beta_n)^2$. Hence, we will assume that for large N, the calculated spectral components (Equation 27) can be treated as independent $\Delta t\beta_n\chi_2^2/2$ random variables with $\mathcal{E}\{C_n\} = \Delta t\beta_n$ and $\mathrm{var}(C_n) = (\Delta t\beta_n)^2$.

Again we see that merely increasing the length of the time series does not reduce the variance. We can, however, make the variance small by averaging a number of the independent spectral components. An obvious way to accomplish this is to use spectral components obtained from each of M windowed segments of a measured time series, i.e.,

$$C_n^{(\ell)} = \frac{\Delta t}{N}\left|\sum_{k=0}^{N-1} w(k/N)\xi_{k+(\ell-1)N}e^{-i2\pi nk/N}\right|^2 \quad \ell = 1, \ldots, M.$$

The asymptotic statistics of this average are easily calculated. First we note that

$$\hat{C}_n^{(M)} = \frac{1}{M}\sum_{\ell=1}^{M} C_n^{(\ell)} = \frac{\Delta t\beta_n}{2M}\sum_{\ell=1}^{2M} u_\ell^2$$

where the u_ℓ are $\mathrm{N}(0,1)$. The sum of the u_ℓ^2 is a χ_{2M}^2 variate with $\mathcal{E}\{\chi_{2M}^2\} = 2M$, $\mathrm{var}(\chi_{2M}^2) = 4M$ and probability distribution

$$\mathcal{P}\{0 \le \chi_{2M}^2 \le x\} = P(x|\,2M) = \frac{1}{2^M(M-1)!}\int_0^x t^{M-1}e^{-t/2}\mathrm{d}t.$$

It follows immediately that the distribution of $\hat{C}_n^{(M)}$ is given by

$$\mathcal{P}\{0 \le \hat{C}_n^{(M)} \le x\} = P\left(\frac{2M}{\Delta t\beta_n}\,x\,\middle|\,2M\right).$$

For $2M > 30$, we can use the normal approximation

$$P(y|\, n) \approx \Phi\left(\frac{(y/n)^{1/3} - (1 - 2/9n)}{\sqrt{2/9n}}\right)$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \mathrm{d}t.$$

Note that the asymptotic estimate of each spectral component $\hat{C}_n^{(M)}$ is just an estimate of the mean as described in Section 3. Furthermore, the samples are independent and we can use the results presented in Section 3.3.1. The expected value of the spectral component estimate is

$$\mathcal{E}\left\{\hat{C}_n^{(M)}\right\} = \Delta t \beta_n = \Delta t \Gamma_n \int w^2(t) \mathrm{d}t.$$

This estimate is biased. The contribution of the window can be removed by normalization, in which case the expected value of the estimate is simply equal to the discrete Fourier transform of the covariance function $\gamma(u) = \mathrm{cov}(\xi(t), \xi(t+u))$ using N samples and a time increment Δt. Because the time series must satisfy a mixing condition [8], the expected value of each component can be made to approach the Fourier transform of the continuous covariance function by increasing the number of samples N and decreasing the time increment.

The variance of the estimate is

$$\mathrm{var}(\hat{C}_n^{(M)}) = \frac{(\Delta t \beta_n)^2}{M}.$$

In practical applications, we can use Equation 8 to obtain an approximation for the variance, i.e.,

$$\mathrm{var}(\hat{C}_n^{(M)}) \approx \frac{1}{M-1} \sum_{\ell=1}^{M} (C_n^{(\ell)} - \hat{C}_n^{(M)})^2.$$

Since N is large, it is a straightforward matter to calculate confidence intervals using the normal distribution approximation (as described in Section 3.1.4).

6. CONCLUDING REMARKS

We have described some of the statistical uncertainties that arise when measured data are used to characterize radio noise and interference. Mathematical expressions for uncertainties in estimates of the mean, statistical distribution functions (viz. the *amplitude probability distribution*), covariance functions and power spectral density functions are presented. In addition to calculating uncertainties, the material presented in this report can be used to develop measurement strategies that minimize data sampling errors.

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15. ABSTRACT

The study of noise and interference in the radio environment is essential to the development of efficacious communications systems. Many of the characteristics of radio noise and interference of interest to radio system designers can be expressed in terms of first- and second-order statistics. These statistics are necessarily calculated using measured data. In this report, we discuss uncertainties that arise when measured data sets are used to calculate statistics of radio noise and interference.

16. Key Words

amplitude probability distribution; average power; measurement uncertainties; power spectrum; radio interference; radio noise

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