On the Definition and Estimation of Spectrum Occupancy

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Abstract—Spectrum occupancy for channels and bands of similar channels is defined. A distinction is made between transmission occupancy and message occupancy. The measurement time required to determine if a given channel is occupied is considered as well as the time required to estimate the degree of transmission occupancy with a given statistical confidence. Nonparametric (distribution-free) statistical techniques are employed to obtain this estimate and to determine the sample size required to establish confidence bounds for a set of channel or band transmission occupancy values.

I. INTRODUCTION

THIS PAPER defines spectrum occupancy and addresses the problem of the measurement time required to estimate the occupancy of radio channels and bands to within some accuracy limits with some given degree of confidence. There are four basic questions:

1) What is spectrum occupancy for a channel and for a band of similar channels?

2) What measurement time is required to determine if a given channel is occupied? We might term this "basic detection."

3) What measurement time is required to estimate the degree of occupancy (percent of total time signals exist on some given channel or in a band of similar channels)?

4) What statistical confidence bounds can be placed upon the sample distribution obtained from a set of occupancy estimates for a channel or band of similar channels.

We address the first question from the standpoint of providing an unambiguous definition of spectrum occupancy on a channel-by-channel basis in a manner amenable to statistical description. We address the last three questions using a minimum of assumptions concerning *a priori* knowledge about the actual structure of the messages (signals) on a channel. Nonparametric (distribution-free) statistical methods are employed to the extent possible in order to avoid continual testing of distribution hypotheses.

We start by defining a random process on which we are to perform the statistical analysis. We use this process to define precisely what we are attempting to measure (or estimate), i.e., "spectrum occupancy." The simplest case (independent samples) is treated first. This case is the most efficient in terms of estimation. Sampling rates are considered, and the case of dependent sampling is also treated. Selected channel models are presented which can be used to estimate the degree of dependency in the measurements and how this might vary with sampling rate, transmission lengths, etc. Distributions of channel transmission occupancy values for the same channel sampled at different times are considered, and these results are extended to distributions of channel occupancy for a band of similar channels. Numerical examples are given as appropriate.

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II. DEFINITION OF CHANNEL TRANSMISSION OCCUPANCY

We first define the transmission occupancy of a selected channel as a two-state random process. The first state is labeled "occupied" and is defined as the event that, during an observation, the signal strength at a monitor receiver is above a given threshold. The complementary event is that the signal strength is below this threshold. Because the state of the channel is random, its state at any given measurement time cannot be predicted. However, its state can be described in terms of a probability law. Traditionally, the unconditional probability that a random sample will be above a threshold has been defined as the "occupancy" of the channel. It has also been expressed as a percent.

If we could observe the occupancy pattern of a channel continuously for an entire hour, we could state with zero error the occupancy for that hour. But if we can make only limited observations of the occupancy pattern, then we can only estimate the occupancy for that hour. It is the task of experimental design to obtain as much information as possible concerning a random process (the random process here being the dynamic occupancy pattern) with the minimum of experimental work. If we use a scanning receiver and take some number of observations of a channel during an hour, we base our estimate of the occupancy of the channel on this number of observations.

Several time scales are important for spectrum occupancy measurements with a scanning receiver. The most basic of these is the time of an individual transmission. Ideally, the scanning receiver should sweep rapidly enough so that all signals in the monitored area are observed several times per transmission. Let us use land-mobile radio as an example. It is important to sort out the undesired impulsive noise from the desired land-mobile transmissions, as well as to record all transmissions. Hence, the shortest sampling time of interest is the time interval between consecutive observations of a given channel. The next time scale of interest, after transmission length, is the time of an individual message (which is usually composed of more than one transmission). The next time of interest is the time of sampling of a given channel. Clearly, we must sample rapidly enough and long enough to resolve the variations of interest, if we can. The other scales of interest are related to trends of use of the channels, such as hourly, daily, weekly, seasonal, and annual variations.

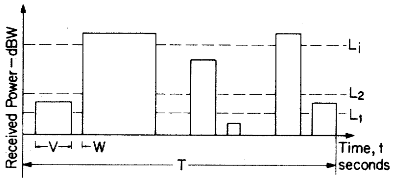

Fig. 1. Transmissions on a channel.

Now, let us define channel transmission occupancy in a rigorous manner. Fig. 1 depicts our arbitrary channel.

In Fig. 1, the random variable V is the transmission length, or more precisely, the length of time a signal continuously exists above some threshold level L_i during the time interval T. The random variable W is the length of time between transmissions, and T is the total time during which measurements of the channel are made. We assume that T is large compared to the mean values of V and W. The random variables V and W have distributions, but we will not specify them.

Consider the two-valued random process: $X(t) = 1$, if signal at time t is above threshold level L_i, and $X(t) = 0$, otherwise.

The process $X(t)$ has "states" 0 and 1. V is the time continuously in state 1, and W is the time continuously in state 0. Let $X(t) = x_i$ so that our sequence of measurements, $x_1, x_2, x_3, \cdots, x_n$ is represented by a sequence of 0's and 1's, where 1 means the detected power in the channel exceeds a given threshold, and 0 means that the detected power in the channel does not exceed that threshold. We are interested in $\beta(L_i, T)$, the fraction of the measurement time T, that $X(t)$ is in state 1, i.e., that the detected power in the channel exceeds threshold level L_i. Since V and W are random variables, $\beta(L_i, T)$ is a random variable which we will call channel transmission occupancy.[1]

We now make two simple assumptions: V and W are independent random variables, and $V + W$ has a continuous distribution.

For our purposes, it suffices to know the behavior of the zero-one process after it has been operating for a long time. Two limit theorems provide very simple answers:

$$\lim_{T\to\infty} P[X(t) = 1] = \frac{E[V]}{E[V] + E[W]} \tag{1}$$

where $E[V]$ is the expectation or mean value of V, etc.

The fraction of time $\beta(L_i, T)$ of the interval 0 to T that the process has the value 1 for threshold level L_i is asymptotically normally distributed; i.e., for every real number x,

$$\lim P\left[\sqrt{T}\frac{\beta(L_i, T) - m}{\sigma} \leqslant X\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X} e^{-y^2/2}\, dy \tag{2}$$

[1] See [1] for a discussion of the difference and relationship between transmission occupancy and message occupancy. Spectrum users are more concerned with fitting into the gaps between messages which may consist of one or more transmissions.

where

$$m = \frac{E[V]}{E[V] + E[W]} \tag{3}$$

and

$$\sigma^2 = \frac{E^2[V]\ \mathrm{var}\ [W] + E^2[W]\ \mathrm{var}\ [V]}{\{E[V] + E[W]\}^3} \tag{4}$$

where var $[V]$ is the variance of V, etc.

The proofs of the above are quite involved,[2] and the details are of no interest to us. Independent of the actual distributions of V and W, we have now defined channel transmission occupancy, $\beta(L_i, T)$ (asymptotically, at least) in terms of the means and variances of V and W. From now on we will be concerned with the determination of the single number m above. We might even call m "the average channel transmission occupancy." Note that by using the artifact $X(t)$, we have been able to use two powerful results for two-valued processes and have specified our statistical estimation problem as the problem of estimating m above. Note further that $m = P[X(t) = 1] = p$. Therefore, terming p the probability of success (i.e., obtaining a 1, or a measurement of signal above our threshold), we have framed our problem to the case of estimating the probability of success in Bernoulli trials, since our sequence of measurements has been represented as a sequence of 0's and 1's. For the case of independent trials, this is the well-known case of binomial sampling, for which many results are available ([4] and the references therein).

III. INDEPENDENT SAMPLING

Our problem, that of estimating p, the probability of success, in Bernoulli trials, has been treated in detail by Crow [4] for independent trials. We start here by considering the effect of the actual measurement time. We seldom have truly "instantaneous" measurements of the energy in a channel. We require a small measurement time t_0, and we will demonstrate that this will not have an effect on our basic results. Suppose that N transmissions occur during time period T and that they occur via the Poisson process, i.e.,

$$p(N) = \frac{(aT)^N}{N!} e^{-aT} \tag{5}$$

where a is the mean number of transmissions per unit time (seconds). The Poisson process for the occurrence of "events" arises from very basic assumptions. The assumption (5) means that the number of changes of state in $X(t)$ or transversals (0 to 1 or 1 to 0), k, in time period T is given by

$$p(k, T) = \frac{(bT)^k}{k!} e^{-bT} \tag{6}$$

where $b = 2a$. In order for the scanning receiver system to

[2] For proof of (1) see Parzen [2], and for (2) see Renyi [3].

detect a signal, where t_0 is the measurement time, we have that P (detection) = $P[X(t) = 1] \times P$ [zero transversals in times t_0]. So

$$P[\text{detection}] = \frac{E[V]}{E[V] + E[W]} e^{-2at_0}. \quad (7)$$

As an example, suppose the average transmission length is 10 s = $E[V]$, the average number of transmissions per hour is 20, so $E[V + W]$ = 3600/20 s, then $p = 10 \times 20/3600 = 0.0556$ (5.56 percent). Let $t_0 = 1.0$ ms[3] so that the factor due to t_0 is $\exp[-2 \times 20/3600 \times 10^{-3}] = 0.99998$. In this example, P [detection] and p are, for all practical purposes, the same.

In the above, it would appear that we have demonstrated the obvious. However, in such problems, small things like t_0 can sometimes have big effects, and it is not safe to neglect them out of hand.

We now consider our second question regarding "basic detection." The probability of m detections in M scans (or measurements) is given by

$$\binom{M}{m} (p)^m (1-p)^{M-m} \quad (8)$$

so the probability of at least one detection (success) in M trials is

$$P = 1 - (1-p)^M. \quad (9)$$

For a 0.99 probability (say) of at least one success

$$0.99 = 1 - (1-p)^M \quad (10)$$

or for M the required number of trials (measurements),

$$M = \frac{\log[1-P]}{\log[1-p]} \quad (11)$$

where P is our required confidence (0.99 above). Table I illustrates the results obtained from (11).

For example, the above tells us that if we take 4603 measurements (on a given channel) and detect no signals, then we are 99 percent confident (99 times out of 100) that occupancy is below 0.1 percent. Remember this is for *independent* samples.

[3]Modern spectrum monitoring systems [5]-[7] can make many measurements in a given channel in 1 ms. The Federal Communications Commission (FCC) system van [5] typically uses 0.5 ms when sampling the nongovernment land-mobile band and the Office of Telecommunications (OT) van [6] uses 0.8 ms when sampling bands in the frequency range 100–500 MHz. The OT system currently takes 40 measurements in 0.8 ms and then uses the minimum of these measurements for the channel signal level. This technique is effective in reducing erroneous measurements due to impulsive noise. The FCC system software considers three adjacent scans and states that if a signal having an amplitude at least 6 dB above threshold is received on a particular frequency preceded and followed by signals on the same frequency having amplitudes near threshold, the higher level signal is impulse noise. Samples considered to be impulse noise are not used in the occupancy computation.

TABLE I
TRIALS (MEASUREMENTS) REQUIRED TO DETERMINE WITH 99-PERCENT CONFIDENCE IF A GIVEN CHANNEL IS OCCUPIED (INDEPENDENT SAMPLES)

Occupancy (p x 100)%	M (99% confidence)
50	7
20	21
10	44
1	459
0.1	4,603
0.01	46,050

We now give a procedure for obtaining confidence limits for the estimation of the probability of success p. We use procedures that are designed to be quite accurate for $p \leqslant 0.1$ (i.e., channel occupancy $\leqslant$ 10 percent). The following results are valid for any value of p, but for $p > 0.1$, other techniques are available which give somewhat "tighter" confidence limits [8]. For any fixed measurement time (number of samples), the smaller p is, the less accurate our estimate of p is, so we are primarily interested in confidence limits for small p. The situation is identical for large p; in fact, the estimation procedures are completely symmetrical about $p = 0.5$. For $p > 0.5$, the results below are applied to the complimentary event $q(q = 1 - p)$. This then, of course, establishes confidence limits for p, $p > 0.5$.

A confidence interval (or set of confidence intervals) for p is a set of random intervals such that, whatever p is, the random interval covers p with a probability at least equal to a prescribed number called the confidence level. The confidence level is denoted as $1 - 2\alpha$; that is, for a confidence level of 0.9 (90 percent), $\alpha = 0.05$. The occurrence of successes (1's) in our sequence of measurements (n samples) is governed, for independent samples, by the binomial distribution. For large n and small p (or small q), the case we have here, it is well known that the Poisson distribution provides an excellent approximation.

If we denote the number of successes by c, and the total number of samples by n, then the "best" (in terms of unbiassedness and efficiency) point estimate of p is simply

$$\hat{p} = \frac{c}{n}. \quad (12)$$

To obtain confidence limits for this estimate we use the upper (U) and lower (L) confidence factors given by Crow and Gardner [8]:

$$\left.\begin{matrix} U \\ L \end{matrix}\right\} = c \pm \frac{1}{2} + \frac{3}{8} u_\alpha^2 \pm u_\alpha \sqrt{c \pm \frac{1}{2} + \frac{1}{8} u_\alpha^2} \quad (13)$$

where u_α is the upper 100 $\alpha/2$ percentage point of the normal distribution of mean 0 and variance 1.

Values of u_α are given in Table II.

If we let p_U and p_L denote the upper and lower confidence limits for p, and U and L the corresponding limits for the Poisson mean [from (13) above], then Anderson and Burstein [9], [10] have given simple but accurate confidence limits

TABLE II
CONFIDENCE LIMITS (AND LEVEL OF SIGNIFICANCE) VERSUS u_α

Confidence	α	u_α
80%	0.1	1.282
90%	0.05	1.645
95%	0.025	1.960
99%	0.005	2.576

for p,

$$p_U = \frac{U}{n + (U - c)/2}$$

$$p_L = \frac{L}{n - (c - 1 - L)/2}. \tag{14}$$

As an example, suppose $n = 4000$ measurements and c was 80 successes, then $\hat{p} = 80/4000 = 0.02$, or we would say 2 percent channel occupancy. From (13), at the 90 percent confidence level, $U = 96.3$ and $L = 65.8$, so, from (14), $p_U = 0.0240$ and $p_L = 0.0165$. That is we are 90 percent confident that channel occupancy is between 1.65 and 2.40 percent. At the 99 percent confidence level, $U = 106.2$ and $L = 58.90$, so $p_U = 0.0266$ and $p_L = 0.0148$. If $n = 400$ and $c = 8, \hat{p} = 0.02$, but at the 90 percent confidence level $p_U = 0.0357$ and $p_L = 0.00981$. The accuracy of our estimate depends on the number of successes, c.

Rather than an absolute confidence interval as above, we are probably more concerned with relative accuracy, especially with p unknown, in deciding how long we must measure. A percent half-length is given, for n relatively large, by

$$\text{percent}(\%)\ \text{half-length} = \frac{U - L}{2c} \times 100. \tag{15}$$

The percent half-length tells us (at a given confidence level) our relative accuracy. In the above numerical example, at the 90 percent confidence level, we have, for $c = 80$, $(U - L)/160 = 0.19$, or a relative accuracy of ±19 percent and for $c = 8$, $(U - L)/16 = 0.656$ or a relative accuracy of ±66 percent. The results given in (13) and (15) have been used to prepare Fig. 2. This figure gives the number of successes c required to achieve a given percent half-length of accuracy. For example, to achieve ±10 percent relative accuracy at the 95 percent confidence level, we require about 390 successes. Or, for a channel occupancy of, say 1 percent, $c/n = 0.01$, or $n = 390\ (100) = 39\,000$ measurements required. Fig. 3 shows, for various relative accuracies, the number of required measurements, n, for a given channel transmission occupancy.

All the above assumed independent samples, and it makes no difference whether we take one sample per year or one sample per second. However, since our actual sampling interval is less than a typical average transmission length, we can be quite confident that we will *not* have independent samples for any reasonably dense sampling scheme. This more realistic situation of dependent sampling is covered in the next section.

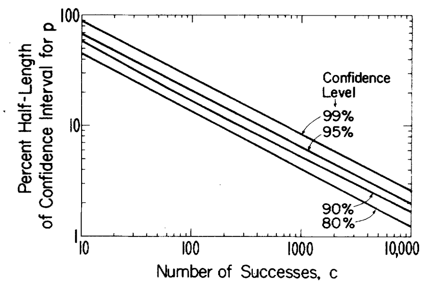

Fig. 2. Relative precision in estimating p from large samples for independent sampling.

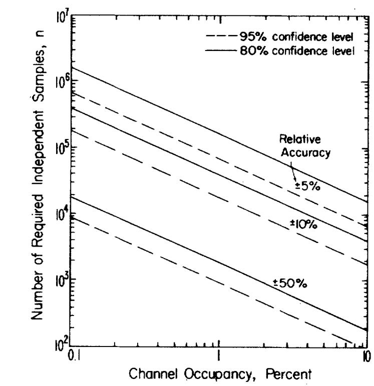

Fig. 3. Number of required independent samples versus channel occupancy.

Before proceeding however, we here mention a test to determine independence. Probably one of the easiest, and best known, tests for independent samples is the run test which is based on the *total number of runs* in a sequence of measurements. A sequence of k identical symbols that is preceded and followed by a different symbol or no symbol is called a run of length k. For example, consider the sequence

000111011010110.

There are five runs of length 1, 2 runs of length 2, and 2 runs of length 3, for a total of nine runs. The total number of runs, μ, in a sequence of independent samples is influenced by the number of 0's, N_0, and the number of 1's, N_1. For a large number of samples, the normal approximation of the distribution of μ is quite satisfactory:

$$E[\mu] = \frac{2N_0N_1}{N_0 + N_1} + 1 \tag{16}$$

and

$$\text{var}\ [\mu] = \frac{2N_0N_1(2N_0N_1 - N_0 - N_1)}{(N_0 + N_1)(N_0 + N_1 - 1)}.$$

If the observed number of runs is *less than*

$$E[\mu] = u_\alpha \sqrt{\text{var}\ [\mu]} \tag{17}$$

we reject the hypothesis of independence at the α significance level. Since here we use a one-sided test, μ_α is the

100α percentage point of the normal distribution of mean 0 and variance 1. For example, if the observed number of runs is less than $E[\mu] + 2.326\sqrt{\text{var}\ [\mu]}$, we reject the hypothesis of independence at the 1 percent significance level ($\alpha = 0.01$). That is we are 99 percent sure that the samples are dependent.

IV. DEPENDENT SAMPLING

In this section we will represent our sequence of 0's and 1's by a first-order Markov chain. Testing whether a sequence can be represented by a Markov chain of first order, or any order for that matter, is a special case of the problem of testing the goodness of fit of a general Markov chain, which has been discussed extensively in the statistical literature. For a summary of such tests see Crow and Miles [11]. Many more complicated mathematical models have been suggested, but for situations of interest to us here, the complex results from such complex models differ little from first-order Markov results.

By a first-order Markov chain, we mean that the probability of success on the ith trial depends on what happened on the $i-1$th trial, but not on the $i-2$nd trial, etc. We assume that the Markov process has been operating for a long time and has achieved steady-state conditions. As before, we have $P[x_i = 1] = p$, and we are interested in estimating p. We have two constant parameters

$$p = P[x_i = 1]$$
$$\lambda = P[x_i = 1 \mid x_{i-1} = 1] \tag{18}$$

where $P[|]$ indicates the conditional probability of the event given first occuring, given that the event given second has occurred. We have the following

$$P[x_i = 0] = 1 - p = q$$
$$P[x_i = 0 \mid x_i = 1] = 1 - \lambda. \tag{19}$$

The second relation in (19) simply says that the process must go from state 1 either to state 1 (probability λ) or to state 0 (probability $1-\lambda$). If we let

$$\phi_{k1} = P(x_i = k \mid x_{i-1} = 1), \qquad k, 1 = 0, 1 \tag{20}$$

then the steady-state equations for our process are

$$q = q\phi_{00} + p\phi_{01}$$
$$p = q\phi_{10} + p\phi_{11}. \tag{21}$$

Solving, we obtain

$$\phi_{00} = P[x_i = 0 \mid x_{i-1} = 0] = \frac{1 - 2p + \lambda p}{q}$$
$$\phi_{01} = P[x_i = 1 \mid x_{i-1} = 0] = \frac{p(1-\lambda)}{q}. \tag{22}$$

The four ϕ_{k1}'s are termed the "transition probabilities." In the above, the probability properties of an entire sequence of n samples are determined by p, λ, and n. Also, $\lambda = p$ means the samples are independent, and $\lambda > p$ means the 1's and 0's cluster. Our problem is now to estimate p for a given λ or to estimate both p and λ. The larger λ is, the more dependent our samples are.

As before, we first consider the question of "basic detection," and we let P denote the probability of at least one success in M trials [and p is the probability of success (signal)]. Let $n_M(1)$ be the number of 1's in M samples. Then

$$P = \Pr\ [n_M(1) \geqslant 1] = 1 - P_r[n_M(1) = 0]$$
$$P = 1 - \Pr\ [x_1 = 0] P_r[\text{next } M-1 \text{ samples are zero}]$$
$$P = 1 - \Pr\ [x_1 = 0]\, [p(x_i = 0 \mid x_{i-1} = 0)]^{M-1}$$

or

$$1 - P = q\left[\frac{1 - 2p + \lambda p}{q}\right]^{M-1}. \tag{23}$$

Solving for M, we have

$$M = \frac{\log(1-P) - \log(1-p)}{\log \dfrac{(1 - 2p + \lambda p)}{1-p}} + 1. \tag{24}$$

Previously, for independent samples, we saw that at the 99 percent confidence level and for 1 percent occupancy, M was 459. Suppose $\lambda = 0.6$. Now for $P = 0.99$, $p = 0.01$ as before, from (24), $M = 1136$. That is, now to be 99 percent confident that occupancy is less than 1 percent, we need 1136 measurements of no signal.

Obviously, λ is a function of the transmission length statistics and our sampling rate.

We now consider the problem of estimating p for the case of dependent samples. Klotz [12] derived estimates of p and λ that are consistent and asymptotically normally distributed. They are

$$\hat{p} = c/n \tag{25}$$

and

$$\hat{\lambda} = \frac{1}{2}(c - \hat{p})^{-1}[r - c + t + (2c - t - 1)\hat{p} + \{[r - c + t + (2c - t - 1)\hat{p}]^2 + 4r(c - \hat{p})(1 - 2\hat{p})\}^{1/2}] \tag{26}$$

where

$$r = \sum_{i=2}^{n} x_{i-1}x_i, \quad c = \sum_{i=1}^{n} x_i, \quad t = x_1 + x_n. \tag{27}$$

An intuitive estimate of λ, not making full use of the data, is the relative frequency estimate.

$$\lambda^* = \frac{r}{c - \hat{p}}. \tag{28}$$

The simplest and most appropriate confidence limits for p for cases of interest here ($p \leqslant 0.1$, large sample size) have been developed by Crow and Miles [11] as a modification to the Anderson-Burstein limits [9], [10] for independent samples discussed in Section III:

$$p_U = \hat{p} + (p_{UI} - \hat{p})\left(\frac{1+\hat{\rho}}{1-\hat{\rho}}\right)^{1/2}$$

$$p_L = \hat{p} - (\hat{p} - p_{LI})\left(\frac{1+\hat{\rho}}{1-\hat{\rho}}\right)^{1/2} \tag{29}$$

where p_{UI} and p_{LI} are the Anderson-Burstein limits for independent samples, and

$$\hat{\rho} = \frac{\hat{\lambda} - \hat{p}}{1 - \hat{p}}. \tag{30}$$

If the p_L calculated from (29) turns out to be negative, then (29) should be replaced by

$$p_U = (p_{UI} - p_{LI})\left(\frac{1+\hat{\rho}}{1-\hat{\rho}}\right)^{1/2}, \qquad p_L = 0. \tag{31}$$

In terms of relative precision, for large n, the percent half-length is, from (29) or (31).,

$$\frac{U-L}{2c} \times 100 = \left(\frac{UI - LI}{2c}\right)\left(\frac{1+\hat{\rho}}{1-\hat{\rho}}\right)^{1/2} \times 100 \tag{32}$$

where UI and LI now denote the upper and lower confidence factors for independent samples [from (13)].

In order to use the above, we need an estimate of λ. This can be obtained from the actual measurements via (26) or (28) or calculated from some assumed channel model.

Suppose we are given the maximum value of λ expected, $\lambda_{\max}$. If we let c_{ind} be the number of successes required to achieve a given relative accuracy at some confidence level for independent samples (Fig. 2), then, to achieve the same prescribed precision approximately, the number of successes required for dependent samples is given in [11] by:

$$c = c_{\text{ind}}\left(\frac{1+\lambda_{\max}}{1-\lambda_{\max}}\right). \tag{33}$$

The next question is that of stationarity. If it is believed that p and/or λ may change during the acquisition of data, the data should be separated into subsamples considered to be homogeneous and tested for differences between the subsamples. Crow and Miles [11] gave procedures for doing this.

An Example

Suppose we have 40 transmissions per hour and our average transmission length is 6 s, then $E[V] = 6, E[W] = 84$, and transmission occupancy $= E[V]/E[V + W] = 6/90 = 6.67$ percent. Suppose we want to estimate p to ±10 percent relative accuracy at the 95 percent confidence level. For independent samples, Fig. 2 tells us that we need 390 successes, or, for the above situation, $n = 390/0.0667 = 5850$ measurements. Let us denote our sampling time interval of τ and let $\tau \approx 4$ s.[4] That is, any given channel has a measurement taken every 4 s. Suppose the transmission lengths above are exponentially distributed with mean value 6 s. A recent study by Lauber and Macklon [13] shows that the usually assumed exponential distribution for transmission lengths is a reasonable assumption—at least for some channels. We use this assumption here, however, only to get an estimate of λ. We have

$$p_V(v) = ae^{-av}, \qquad a = 1/6$$

$$P[x_i = 1 \mid x_{i-1} = 1] = P[V \geqslant \tau] = \int_{\tau}^{\infty} ae^{-ax}\, dx = e^{-a\tau}.$$

The above result is independent of where, during a message, x_{i-1} occurs. This is due to the "memorylessness" property of the exponential distribution. The exponential distribution is the only distribution that has this property. [Note that, if we know that the message lengths are exponentially distributed, then we can estimate the average message length from an estimate of λ via (26) or (28). That is, $\hat{a} = -(\ln \hat{\lambda})/\tau]$. For $\tau = 4$ s then

$$\lambda = e^{-4/6} = 0.513.$$

If we decide this value of λ is the maximum we expect, then the required number of successes to achieve the same relative accuracy as above (±10 percent, 95 percent confidence) is

$$c = 390\left(\frac{1+0.513}{1-0.513}\right) = 1212$$

or $n = 1212/0.0667 = 18\,166$. At a measurement every 4 s, then, for independent samples, we require a measurement time of 6.5 h, but for dependent samples ($\lambda = 0.513$), we need a measurement of time of 20.2 h. Many channels of interest may not be stationary over such a long interval because of diurnal variations of channel usage.

Another Example

Suppose we have three bands and we can scan each band in 4 s. Suppose we want ±10 percent relative accuracy at the 95 percent confidence level; then, for independent samples, we require 390 successes. We want to determine channel occupancy for all channels in the three bands with the above relative accuracy down to occupancy of 5 percent. We consider two options:

1) Scan the first band for the required amount of time and then go to the second band, etc.

2) Scan all three bands together.

[4] A current procedure [6], for example, scans a band of channels such that each channel is revisited every 4 s.

Obviously, for independent samples, there is no difference between options 1) and 2). Suppose, now, we assume transmission length statistics as in the above example ($E[V] = 6$ s, etc.) and that this is true for all channels of the three bands. Under option 1),

$$\lambda = e^{-4/6} = 0.513$$

and the number of successes we require is, therefore, 1212. Then, $n = 1212/.05 = 24\ 240$ scans or a measurement time of 27 h/band for a total measurement time of *81 h.*

Under option 2),

$$\lambda = e^{-12/6} = 0.135$$

so, now, the minimum number of successes required is

$$390\left(\frac{1+0.135}{1-0.135}\right) = 512.$$

Then, $n = 512/0.05 = 10\ 240$ scans, or a total measurement time of $12 \times 10\ 240/3600 =$ *34 hs.*

Suppose we want ±10 percent relative accuracy at the 95 percent confidence level, occupancy down to 0.1 percent ($p = 0.001$). For option 1), $\tau = 4$ s and $\lambda = 0.513$. We require 1212 successes:

$$n = \frac{512}{0.001} = 1\ 212\ 000 \text{ scans or } 1347 \text{ h/channel}$$

or

$$3 \times 1347 = \textit{4040 h} \text{ total measurement time.}$$

For option 2),

$$n = \frac{512}{0.001} = 512\ 000 \text{ scans}$$

so total measurement time is

$$\frac{12 \times 512\ 000}{3600} = \textit{1707 h.}$$

Suppose now we want down to 0.1 percent ($p = 0.001$) occupancy but ±50 percent relative accuracy at 80 percent confidence level, so

$$c_{\text{ind}} = 9 \text{ samples.}$$

For option 1),

$$c = 9\left(\frac{1+0.513}{1-0.513}\right) = 28$$

$$n = \frac{28}{0.001} = 28\ 000 \text{ scans or } 31 \text{ h/band}$$

total time = 3 X 31 or 93 h.

For option 2),

$$c = 9\left(\frac{1+0.135}{1-0.135}\right) = 12$$

$$n = \frac{12}{0.001} = 12\ 000 \text{ scans}$$

on total time = (12 X 12,000)/3100 = *41 h.*

One possible area of concern, statistically, is whether a little used channel would exhibit the same sort of discipline as a heavily used channel. In a crowded channel, the user is likely to use procedures to minimize actual time on the air (e.g., the "10 code"). These would tend to make transmissions shorter on heavily used channels. On the other hand, the user of a little used channel has no reason to be efficient and may feel more sure of himself if he talks a bit more. Therefore, the transmissions on a little used channel may average several times the length of those on a heavily used channel. Suppose we have $p = 0.001$ to measure, and the average transmission length is now 30 s. Then $\lambda = \exp(-\tau/30)$, so that for $\tau = 4$ s, $\lambda = 0.875$, and for $\tau = 12$ s, $\lambda = 0.670$. For option 1), 95 percent confidence, ±10 percent relative accuracy,

$$c = 390\left(\frac{1+0.875}{1-0.875}\right) = 5859$$

or

$$n = 5\ 859\ 000 \text{ scans per band}$$

so total time = (3 X 4 X 5 859 000)/ 3600 = *19 530 h.*

For option 2), 95 percent confidence, ±10 percent relative accuracy,

$$c = 309\left(\frac{1+0.670}{1-0.670}\right) = 1977$$

or total measurement time = (12 X 1 977 000)/3600 = *6590 h.*

The above examples show that if we insist on tight relative accuracy with high confidence for small p, the required measurement times can quickly become quite large.

The Example of Section III [Following (14)] for Dependent Samples

As in Section III, suppose n was 4000 measurements and c was 80 successes. We saw that the $\hat{p} = 0.02$ and that at the 90 percent confidence level and for independent samples, $p_{UI} = 0.0240$ and $p_{LI} = 0.0165$. Suppose now we know that $\hat{\lambda} = 0.6$. Then, from (29) and (30) we have

$$\hat{p} = \frac{0.6 - 0.02}{1 - 0.02} - 0.592$$

$$p_U = 0.02 + (0.0240 - 0.02)\left(\frac{1+0.592}{1-0.592}\right)^{1/2}$$

$$p_U = 0.0279$$

$$p_L = 0.02 - (0.02 - 0.0165)\left(\frac{1 + 0.592}{1 - 0.592}\right)^{1/2}$$

$$p_L = 0.0131.$$

Or, we are 90 percent confident that p is between 1.31 and 2.79 percent. For independent samples, the relative accuracy was ±19 percent with dependent samples from (32), the relative accuracy is

$$\pm 19\left(\frac{1 + 0.592}{1 - 0.592}\right)^{1/2} = \pm 37.5 \text{ percent.}$$

For further examples and results, see Appendix I. An example of data obtained in two channels during the same hour are examined in Appendix II to illustrate the lack of stationarity which can be encountered in practice.

V. DISTRIBUTIONS OF CHANNEL OCCUPANCY VALUES

The previous sections have defined channel transmission occupancy as a random variable and have established confidence bounds on individual occupancy estimates obtained by sampling a channel through the use of distribution-free (nonparametric) statistics. This single value, m, is applicable only for time periods for which the channel statistics remain stationary. We do not expect stationary statistics over long time periods (daytime versus night-time, Monday versus Sunday, for example). For these long-term variations, we will have a number of occupancy values which we want to use to make statements like: The channel transmission occupancy is greater than 50 percent more than 80 percent of the time, etc. Also, we may want to discuss occupancy for a band of channels and make statements like: 62 percent of the channels in the band have occupancies greater than 40 percent etc. In short, we are interested in distributions of channel transmission occupancy values. This section will consider such distributions and a means of placing confidence limits on such distributions.

The average occupancy m of the hth channel for threshold level L_i over the jth time interval T_j, defined in the preceding sections, forms the basic building block of the cumulative distribution function of channel occupancy over the much longer time interval T_k. Over a long time interval T_k, we have η values of occupancy for a channel of interest C_h. From the previous sections, we let $m = O(C_h, L_i, T_j, T_k)$ = transmission occupancy. The η values of $O(C_h, L_i, T_j, T_k)$ computed will be rank ordered, normalized to obtain any estimate of the probability of a given occupancy level, and plotted as a function of occupancy (O_c) to generate the cumulative distribution function (CDF) for the interval T_k. The occupancy data for a given threshold over this time interval can be neatly summarized on a diagram as illustrated in Fig. 4. This process can then be repeated for the other thresholds, which can then be plotted on the same diagram. [A corresponding situation is the cumulative distribution of η values of channel occupancy for h channels forming a band of channels].

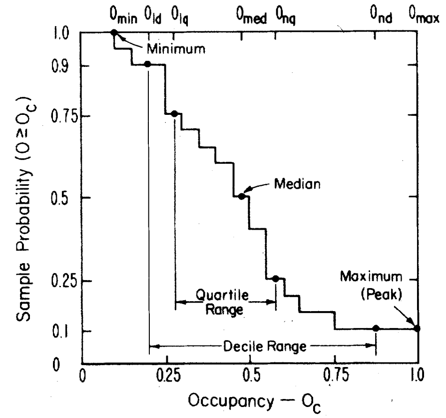

Fig. 4. Hypothetical example of sample cumulative distribution function of channel occupancy for time interval T_k, hth channel, and threshold L_i.

Such a plot is a very useful summary, because one can read directly from it the following occupancy measures for the channel for interval T_k:

peak (0_{max})

median (0_{med})

minimum (0_{min})

quartiles ($0_{uq}, 0_{lq}$)

deciles ($0_{ud}, 0_{ld}$)

quartile range

decile range.

The cumulative distribution functions also provide the vehicle, when coupled with distribution-free statistics, for putting confidence limits on the measured occupancy cumulative distribution.

One nonparametric measure of the deviation between the sample distribution $F_\eta(x)$, and the "true" occupancy distribution, $F(x)$, is the Kolmogorov-Smirnov (KS) statistic, $\sqrt{\eta}$ supremum $|F_\eta(x) - F(x)|$, [14]-[17]. The KS statistic is based on independent samples.

The KS statistic is useful for determining the sample size (η) required to approximate the occupancy cumulative distribution function, $F(x)$, by the sample distribution function $F_\eta(x)$ within a prescribed accuracy with a prescribed level of confidence. For example, if we have the hypothesis that the distributional error never exceeds ±0.15 (±15 percent), then $\eta > 80$ is required for the hypothesis to be accepted at the 95 percent confidence level (see Fig. 6). Tables of this statistic have been given by Massey [14] for levels of significance (α = Type I error) of 0.20, 0.15, 0.10, 0.05, and 0.01, which correspond to confidence levels of 80, 85, 90, 95, and 99 percent, respectively. Figs. 5 and 6 give the half-length confidence interval versus sample size η in terms of the confidence level $(1 - \alpha)$ that the hypothesis is accepted, given that it is true.

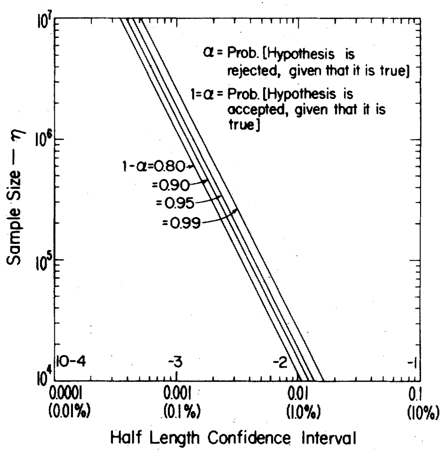

Fig. 5. Number of required samples, η, for a given half-length confidence interval for the distributional error.

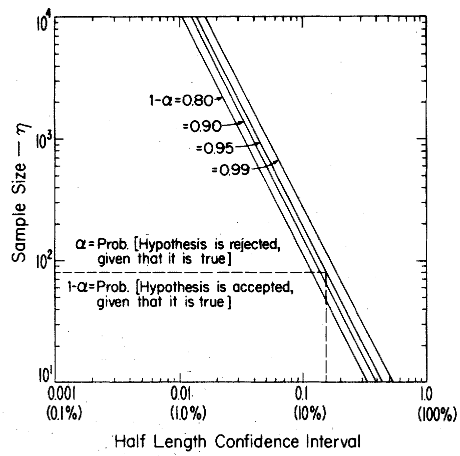

Fig. 6. Number of required samples, η, for a given half-length confidence interval for the distributional error.

When the sample size $\eta > 35$, then Smirnov's [18], [19] asymptotic forms can be used [14]:

$$\epsilon_{\eta,0.95} \cong 1.3581\ \eta^{-1/2}$$

$$\epsilon_{\eta,0.99} \cong 1.6276\ \eta^{-1/2}.$$

The KS statistics are applicable to independent observations with a common (unknown) continuous distribution function $F(x)$. We note in passing that the KS statistics can also be used to test the goodness of fit of a completely specified cumulative distribution [14]-[16].

Let us now consider an example CDF: a land-mobile channel in Chicago [20]. The CDF data shown in Figs. 7 and 8 were obtained for a Special Emergency channel (33.080 MHz) on the same day, at approximately 0700 and 0900. Invalid data consisting of impulsive noise (IN) samples and intermodulation products (IM) were removed, using the methods discussed by McMahon [5].[5] At 0652 hours local time, the

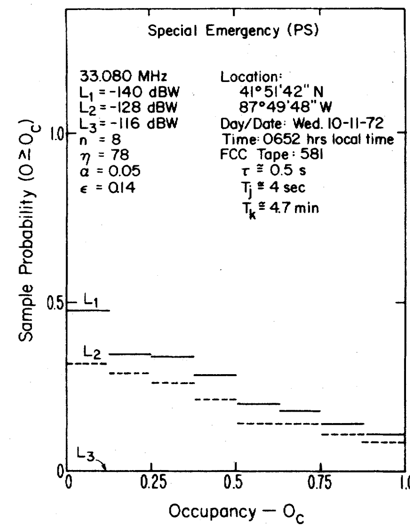

Fig. 7. CDF of lightly loaded 33-MHz channel.

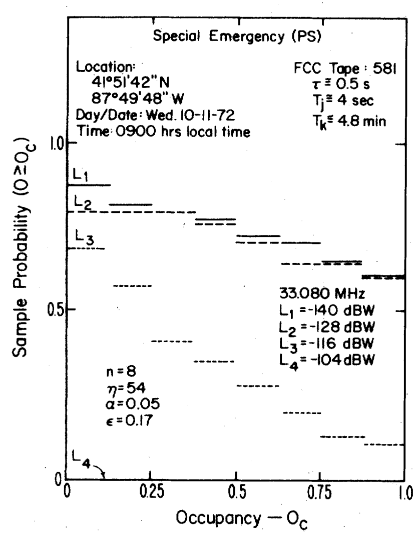

Fig. 8. CDF of relatively heavily loaded 33-MHz channel.

channel was lightly loaded, but at 0900 hours the usage of the channel had greatly increased. This can easily be seen by comparing CDF's.

At 0652 hours, there were 627 valid samples with only one case of IN and six cases of IM. The average transmission time was 2 s. The maximum observed level was − 117 dBW, and the minimum observed level was −163 dBW. The mean transmission occupancy values, m, computed over the entire 4.7-min interval were 26, 24, and 19 percent for −140, −134, and −128-dBW thresholds, respectively; the occupancy was zero for higher thresholds. At 0900 there were only 435 valid samples with three cases in IN and 206 cases of IM. The average transmission time was 1.7 s, and the maximum and minimum observed levels were −110 dBW and −151 dBW, respectively. For L_2 (−128-dBW threshold), the mean transmission occupancy level for the entire 4.8-min interval was 72 percent, and it increased to only 74 percent at L_1 (the −140-dBW threshold). The occupancy level dropped off to 35 percent at L_3 (the −116-dBW threshold) and to only 1 percent at a threshold of −110 dBW.

[5] The IM algorithm used a 3-dB attenuator switched in and out on alternate scans of a given channel (τ = 0.5 ms). This algorithm was conservative in that it tended to reject valid data with contained amplitude fluctuations ⩾ 6 dB (e.g., due to propagation) occurring during 1 ms.

About 15 percent of the data were discarded for the 0652 hours sample and about 30 percent of the 0900 hours sample. Nevertheless, the confidence bound $\epsilon_{\eta,0.95}$ (Smirnov's asymptotic form) increased only from about 0.14 to 0.17, a much better result than if whole blocks of data had been discarded because small sections were invalid.

As noted previously, the methodology for placing confidence bounds on the CDF of a channel can be used for bands of channels. For example, one might compute for the same time interval T_k the median occupancy for each of h channels in a band of similar channels. The CDF for the band occupancy could then be computed, and confidence bounds could be applied using the KS statistics. Such band occupancy statistics could be used to study the time variation of band usage over longer periods of time.

VI. CONCLUSIONS

This paper has defined channel transmission occupancy as a measure of a random process: specifically, the quantity $\beta(L_i, T)$ is the fraction of time T that the received power in the channel exceeds the threshold level L_i. We have shown how, independent of the actual statistical distributions of the individual transmission lengths (V) and the gaps between these transmissions (W), we can describe $\beta(L_i, T)$ asymptotically in terms of the means and variances of V and W over the interval T. The problem of estimating transmission channel occupancy has been reduced to the problem of estimating $p(= m)$, the probability of success in Bernoulli trials, a problem for which many results are in the literature. We have examined how long it takes to measure m to a given accuracy with a given level of confidence for both independent and dependent samples–independent of the actual transmission structure on the channel. We have discussed a method for determining the independence of the samples as well as the time stationarity of the sequence of samples. Once a minimum value of m is set, it is possible, previous to making measurements, to determine the measurement time required to determine m to within a given relative accuracy with a given level of confidence (the degree of dependence between measurement samples, λ, must also be assumed, or estimated). We have explored the tradeoffs between measurement time and measurement accuracy, and shown that if it is desired to get extremely accurate estimates of low-occupancy (or high-occupancy) channels that there may be a problem with lack of sufficient stationarity, due to the required large measurement times.

After discussing the measurement of mean transmission occupancy, m, we showed how a group of m values could be rank-ordered and normalized to form an occupancy cumulative distribution function (CDF) for a channel or for a band of similar channels. Confidence bounds on the CDF were computed using distribution-free techniques that are useful in estimating (before the measurement) the number of m samples required to estimate the occupancy distribution function to a given accuracy with a given level of confidence. Finally, it should be noted that CDF's can be generated on level for a fixed time as well as on time for a fixed level. The same statistical methods would pertain.

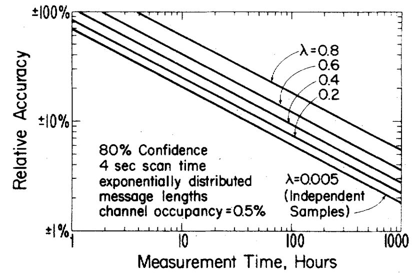

Fig. 9. Measurement time for various degrees of sample dependence, λ.

APPENDIX I

FURTHER RESULTS AND EXAMPLES

We use the results obtained previously to develop various figures that illustrate the kind of results possible. The basic results are those of Fig. 2 and all others proceed directly from the number of successes c. The first of the figures (Fig. 9) gives measurement time versus relative accuracy for various channel occupancy levels. On Fig. 9 we assume 80 percent confidence, and exponentially distributed transmission lengths with a mean of 6 s, so that, as we saw previously, $\lambda = 0.513$. The relative accuracy is given by (32). For example, for 5 percent occupancy, or $\hat{p} = 0.05$, we obtain from (30) that $\hat{\rho} = 0.487$, and then from (32) for ±5 percent relative accuracy,

$$\left(\frac{UI - LI}{2c}\right) = 0.05\left(\frac{1-\hat{\rho}}{1+\hat{\rho}}\right)^{1/2} = 0.0293 \text{ or } 2.93 \text{ percent.}$$

Then from Fig. 2, the required c is 1800; therefore

$$\text{measurement time} = \frac{\left(\frac{1800}{0.05}\right) \times 4 \text{ s}}{3600 \text{ s/h}} = 40.0 \text{ h.}$$

Notice that if we use the approximate upper bound expression (33) with $\lambda_{max} = 0.513$, c is 2019, and the measurement times is 44.9 hr. This example points out that (33) gives an approximate result (upper bound), but (33) is somewhat easier to compute. Fig. 9 was developed via (32).

As we have seen, the required measurement time critically depends on the parameter λ, which, in turn, depends on the transmission length statistics and the rate at which we take measurements on any given channel. Using the assumption of exponentially distributed transmission lengths, for example, we have seen that

$$\lambda = \exp\left[-\frac{\text{measurement interval}}{\text{average transmission length}}\right].$$

Fig. 10 shows the effect of λ on measurement time, where we have chosen 80 percent confidence, channel occupancy of 0.5 percent ($p = 0.005$), exponentially distributed transmissions, and a 4-s scan time.

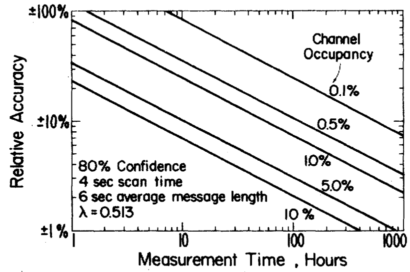

Fig. 10. Measurement time versus relative accuracy for various channel occupancies.

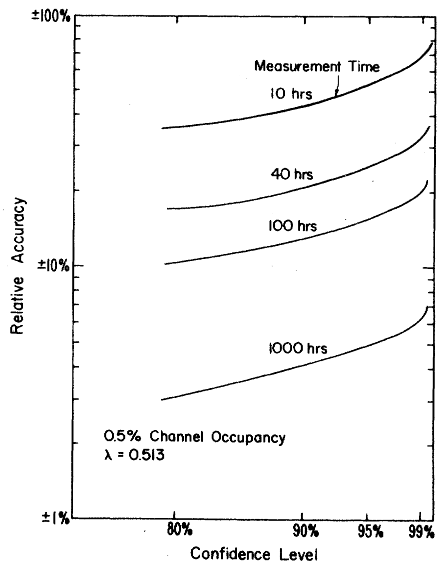

Fig. 11. Relative accuracy versus confidence level for various measurement times.

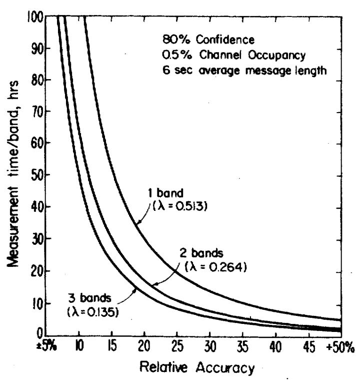

Fig. 12. Measurement time per band versus relative accuracy.

Fig. 11 shows the realtionship between relative accuracy and confidence level for various measurement times. A channel occupancy of 0.5 percent ($p = 0.005$) is used with a scan time of 4 s and exponentially distributed transmissions with 6-s average length.

On Fig. 12, the difference in required measurement time between scanning one band at a time and scanning more than one band is shown. Fig. 12 shows the measurement time *per band*. As before, each band can be scanned in 4s, and exponentially distributed messages of 6-s average length are assumed. Suppose we have three bands and want ±20 percent relative accuracy for channel occupancies down to 0.5 percent. We see from Fig. 12 that, if each band is scanned separately, each band requires 30.5 h of measurement time, or a total of 91.5 h for all three bands. However, if all three are scanned together (so that the sampling interval for each channel is 12 s rather than 4 s) then the measurement time per band is 13.3 h, or a total required time of 39.9 h.

APPENDIX II

EXAMPLES OF CHANNEL TRANSMISSION OCCUPANCY COMPUTED OVER DIFFERENT TIME INTERVALS DURING THE SAME HOUR

Monitoring data [20] were obtained by the FCC for a period of several hours on January 17, 1973 at the City of Chicago South Water Filtration Plant, 3200 East 78th Street (41°45′24″N, 87°32′36″W), as part of a special test. For this test we have data in 5-min simple data sets (i.e., $T_j = 300$ s) for every other 5-min period over several consecutive hours. Each channel was sampled about every 0.5 s for the period between approximately 1530 and 1720 hour local time. Data in two 150-MHz business radio channels from the first hour of observation were reduced as simple and compound data sets to illustrate the compound technique and, more importantly, to show how transmission occupancy computed for a 5-min interval can differ from that computed over longer periods up to an hour.

The two channels studied were 151.865 and 151.895 MHz. Occupancy was computed for seven thresholds for each channel for the first 5 min (simple data sets beginning at 1531 hour), after the invalid data had been removed. The results of these calculations are given in Table III. Also shown in this table are the number of valid samples, the number of invalid data (IN and IM), and the maximum and minimum observed signal levels. Next, a compound data set was formed for each channel by using the first (simple) data set for the given channel and data from the next available 5-min period (which actually began 10 min after the beginning of the first set). The transmission occupancy calculations were then repeated; they are recorded in Table III in the rows labeled 10 min. Finally, all the similar data obtained during the first hour were combined to give transmission occupancy values computed over the equivalent of a 30-min period (see Table III). These tests revealed some interesting things, as discussed below.

The 151.865-MHz channel was essentially unoccupied over the first 5 min; and the same situation existed from the 10-min sample (Table III). A relative increase in occupancy was noted for the 30-min sample. On an absolute scale, the channel was still lightly loaded, but the 5- and 10-min samples (while similar to each other) were not good estimators of the 30-min sample.

In the 151.895-MHz channel, also lightly loaded, a significant increase in activity was observed between the 5- and 10-min samples. The data taken during the 10-min sample were reasonably representative of those taken during the first hour (the 30-min sample in Table III) for thresholds above −140

TABLE III
CHANNEL TRANSMISSION OCCUPANCY FOR DIFFERENT SAMPLE INTERVALS DURING THE SAME HOUR

Frequency (MHz)	Total Interval Scanned (min)	Total Elapsed Time (min)	Mean Transmission Occupancy vs. Threshold (percent)							No. of Samples			Signal (dBW)	
			- 140 dBW	- 134 dBW	- 128 dBW	- 122 dBW	- 116 dBW	- 110 dBW	- 104 dBW	Valid	IN	IM	Maximum	Minimum
151.865	5	5	5.0%	1.0%	0.0%	0.0%	0.0%	0.0%	0.0%	378	4	2	- 127	- 160
151.865	10	15	6.0	1.0	0.5	0.0	0.0	0.0	0.0	722	4	42	- 124	- 161
151.865	30	55	9.6	3.9	2.9	1.4	0.7	0.4	0.0	2093	31	323	- 106	- 161
151.895	5	5	11.0	2.0	1.0	1.0	1.0	1.0	0.0	366	8	10	- 105	- 159
151.895	10	15	11.5	5.0	4.0	3.5	3.0	2.0	0.0	721	18	29	- 105	- 160
151.895	30	55	11.1	4.1	3.9	3.3	2.5	1.4	0.4	2298	42	107	- 104	- 160

dBW. The data taken during the first 5 min were reasonably representative of the first hour, for the −140-dBW threshold (about 20 dB above the level of the minimum observed "signal"). This sample illustrates the need to consider threshold level when evaluating how representative data from a given 5-min interval are for the hour from which they came. One might expect greater similarity of occupancy data on lower thresholds, especially near the noise level.

A conclusion from this brief study of time stationarity (in the loose sense) is that it is important to try different groupings of the same data when evaluating the adequacy of any sampling plan that looks at each channel for only 5 min of each hour. More data should be taken over longer continuous intervals, and attention should be given to the results for channels with greater occupancy than the two cited here.

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